

# Constraint on higher dimensional gravity theory from black holes

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# Higher dimensional BHs

- String theory: BH entropy  $\leftrightarrow$  microstate counting
- Holography: gauge/gravity correspondence
- BHs in brane world scenario
  - production at collider
  - BH spacetime in brane world

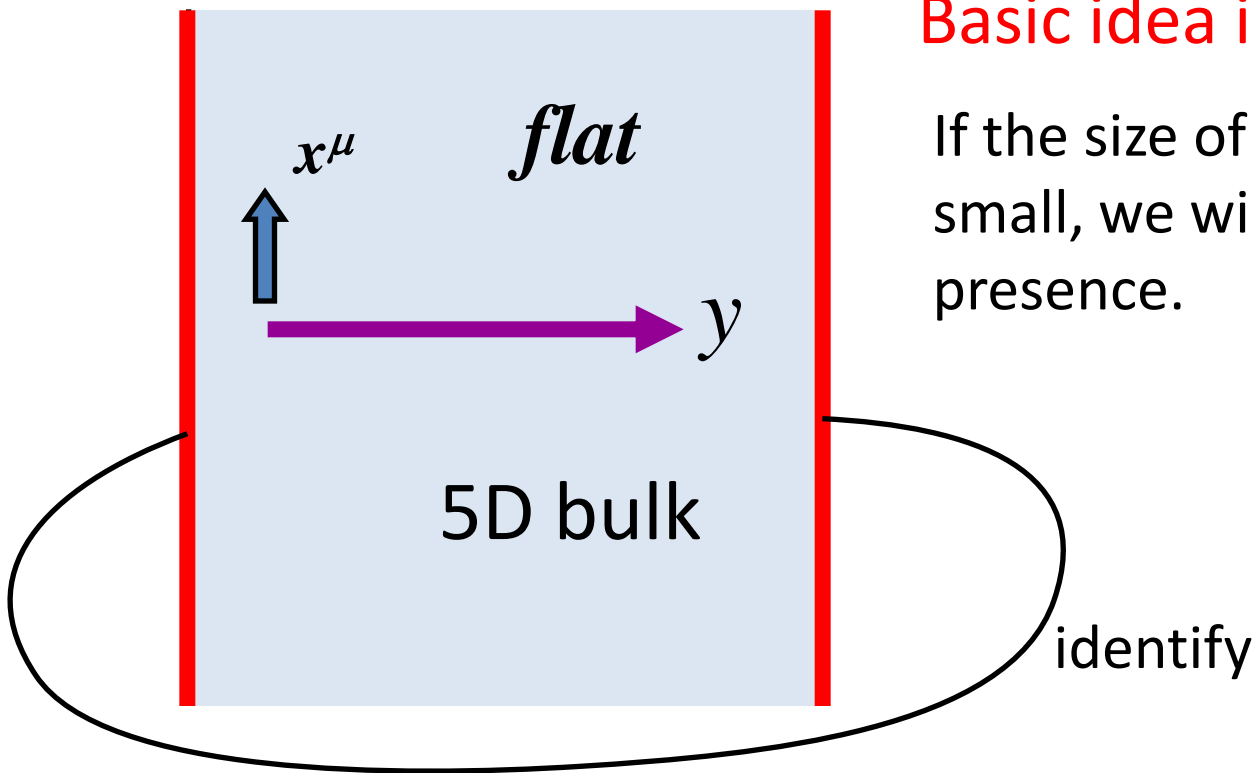
# Contents

- Brane World Scenario
- Black Holes at Collider
  - Shock Wave
  - Quasi Stationary Evaporation
  - Ultra Spinning Black Holes
- Black Holes in Warped Extra-Dimensions

# Compactification

- Fundamental theory of particles
  - Superstring theory (10dim) or M-theory (11dim)
- Our universe is four dimensional

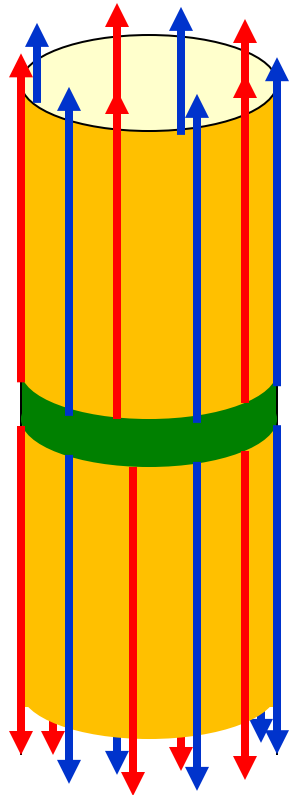
→ Compactification!



Basic idea is:

If the size of extra dimensions is small, we will not sense their presence.

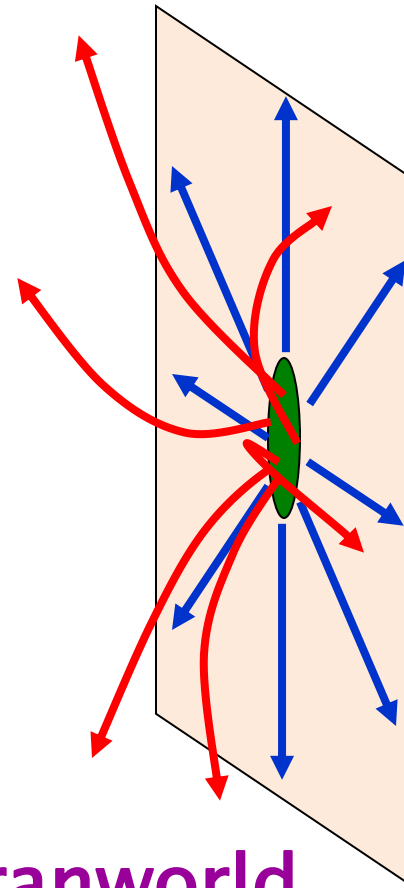
# A new scheme for compactification



Flux lines  
of gravity

Electric  
flux line

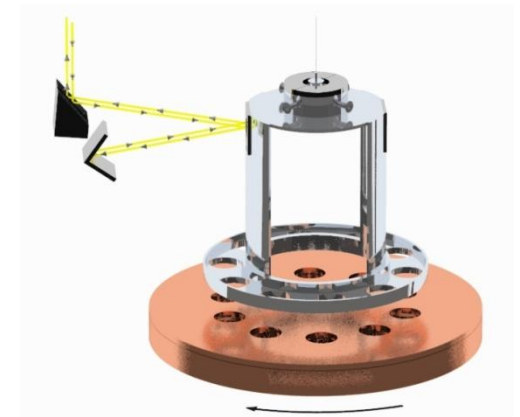
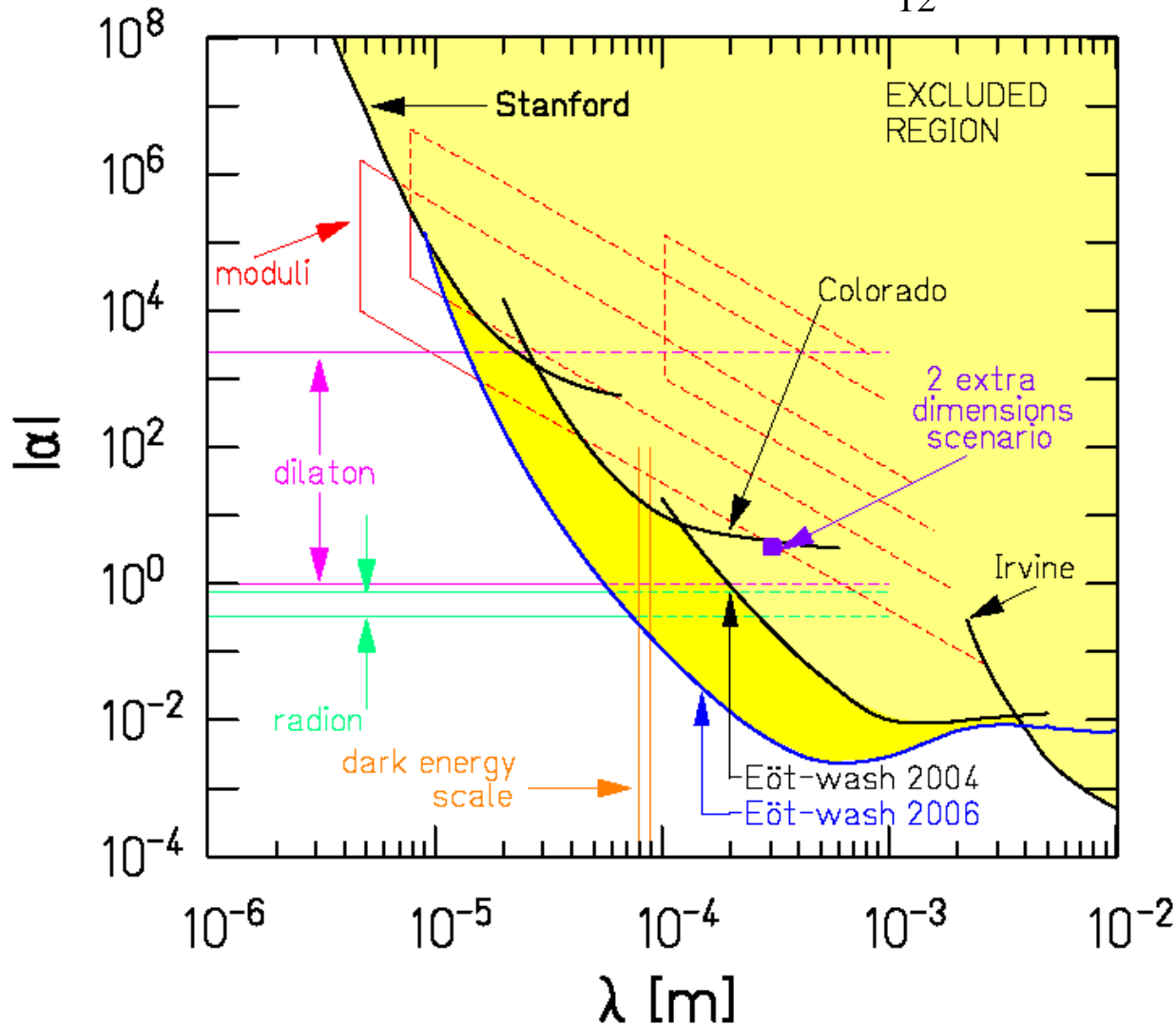
Kaluza-Klein  
compactification



Braneworld  
hierarchy problem  
modified gravity  
and cosmology

# Constraint on deviation from Newton's law

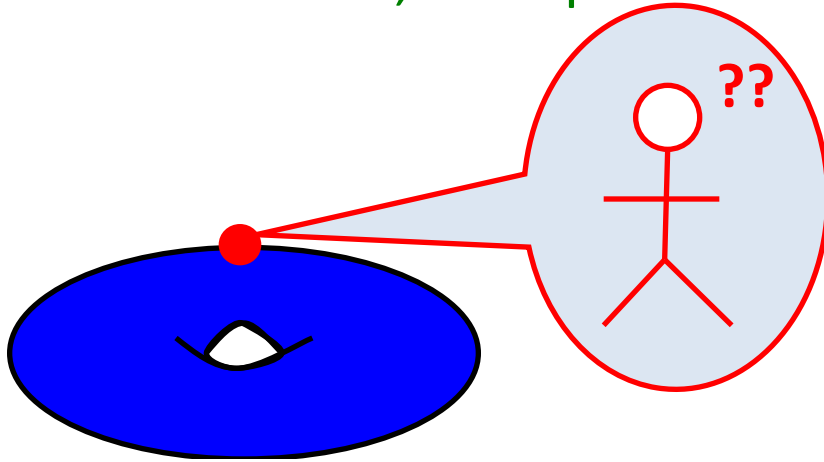
- **Short range force**  $U = -\frac{Gm_1m_2}{r_{12}} [1 + \alpha \exp(-r_{12}/\lambda)]$



Capner et al  
[hep-ph/0611184](https://arxiv.org/abs/hep-ph/0611184)

# Large extra dimensions

Arkani-Hamed, Dimopoulos and Dvali (1998)



$$S = \frac{1}{16\pi G_{n+4}} \int d^4x \int d^n y \sqrt{-g} R$$

Homogeneity in directions of extra dimensions

Minkowski  $\times T^2$

Volume of  $T^2$

$$\rightarrow \frac{V_n}{16\pi G_{n+4}} \int d^4x \sqrt{-g^{(4)}} R^{(4)}$$

Effective Newton constant

$$\sim 1/16\pi G_N$$

$$M_{pl}^2 \approx M_{n+4}^{n+2} \underline{d^n}$$

Length scale of extra dimensions

$n=2,$   
 $M_6$  at electroweak scale  
 $1\text{TeV} = 10^3\text{GeV}$   
 $d=1\text{mm} \approx (10^{-13}\text{GeV})^{-1}$   
 $(10^{19}\text{GeV})^2 \approx (10^3\text{GeV})^4 (10^{-13}\text{GeV})^{-2}$

# Warped extra dimension


Randall Sundrum I model (1999)

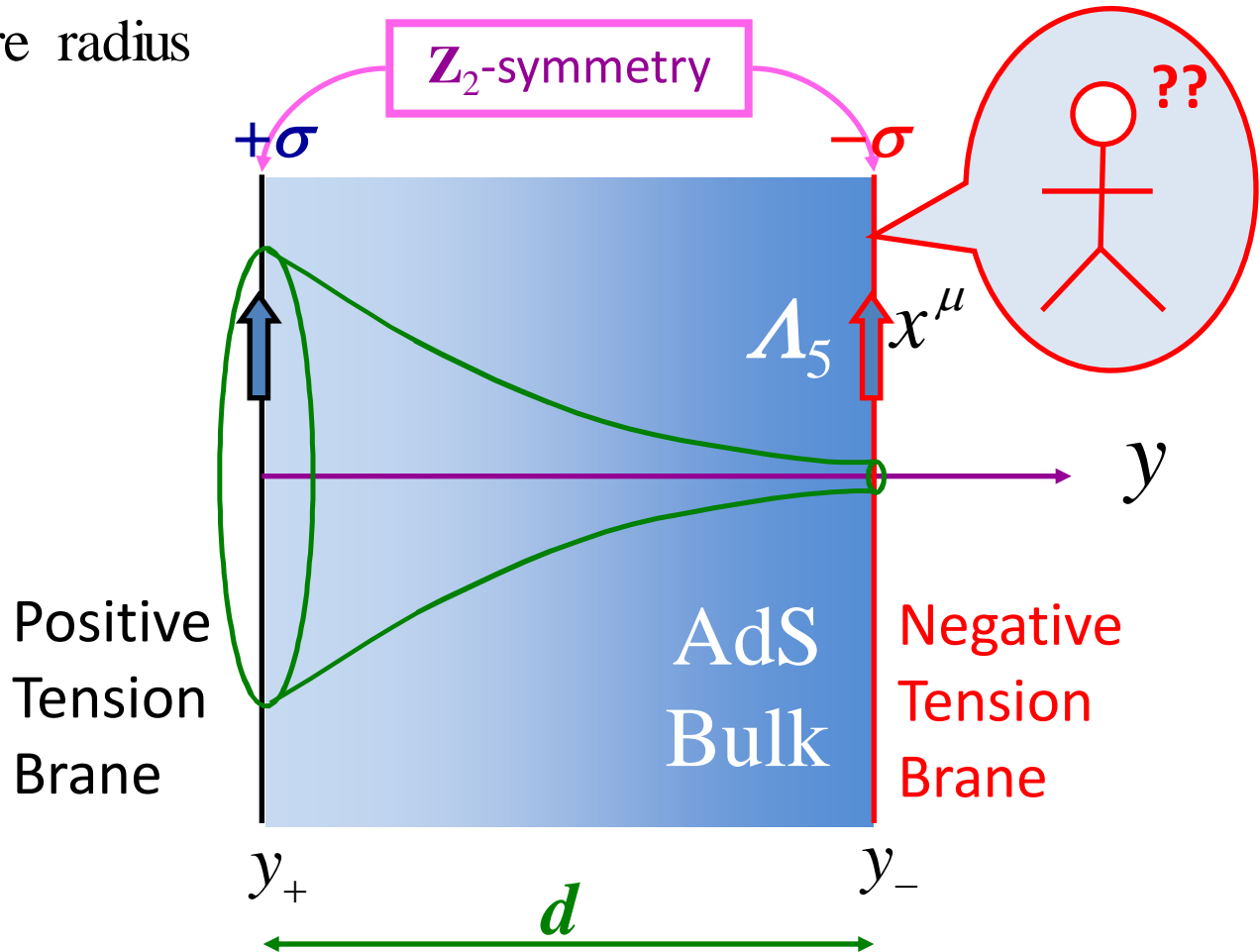
Anti-de Sitter  $ds^2 = dy^2 + e^{-2y/\ell} (\eta_{\mu\nu} dx^\mu dx^\nu)$

$\ell$  : AdS curvature radius

$$\Lambda_5 = -\frac{6}{\ell^2}$$

$$\sigma = \frac{3}{4\pi G_5 \ell}$$

5-d  Gravitational coupling constant



$$ds^2 = dy^2 + e^{-2y/\ell} \left( g_{\mu\nu}^{(4)} dx^\mu dx^\nu \right)$$

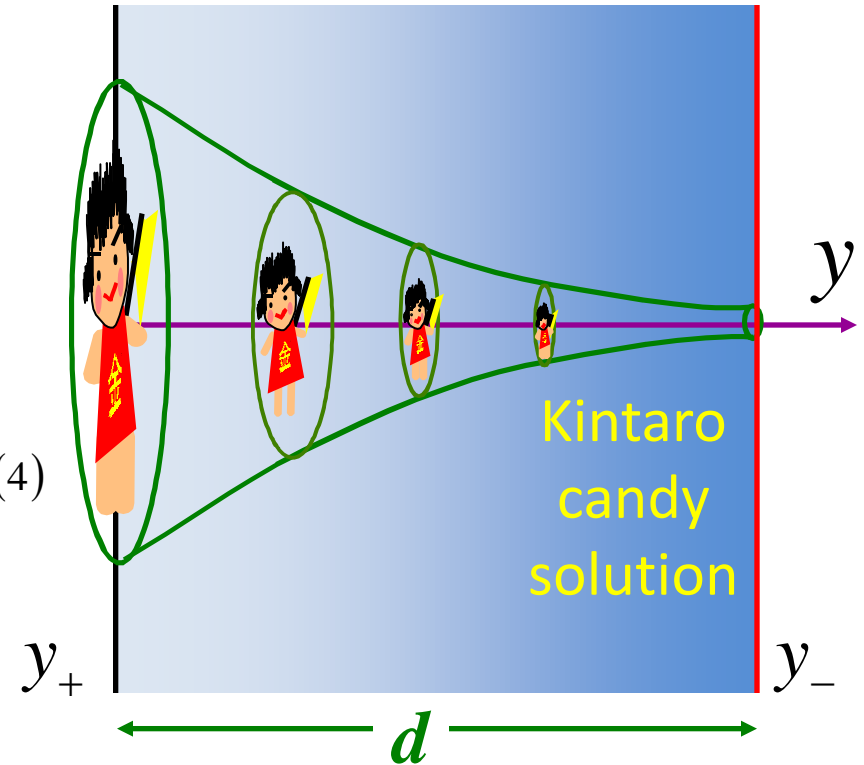
If  $g_{\mu\nu}^{(4)}(x)$  is 4d solution  $\Rightarrow$  the above 5d metric is also solution.

$$S = \frac{M_5^3}{2} \int dy \int d^4 x \sqrt{-g} \{R + 2\Lambda_5\}$$

Substituting Kintaro  
Candy solution

$$S \supset M_5^3 \int_{y_+}^{y_-} dy e^{-2y/\ell} \int d^4 x \sqrt{-g^{(4)}} R^{(4)}$$

$$\frac{\ell M_5^3}{2} \left( e^{-2y_+/\ell} - e^{-2y_-/\ell} \right)$$



Negative tension brane

$$\underbrace{M_{pl}^2}_{(10^{19}\text{GeV})^2} = \left( e^{2d/\ell} - 1 \right) \underbrace{M_5^3 \ell}_{(\text{TeV})^2}$$

for  $d \sim 40$  🌸

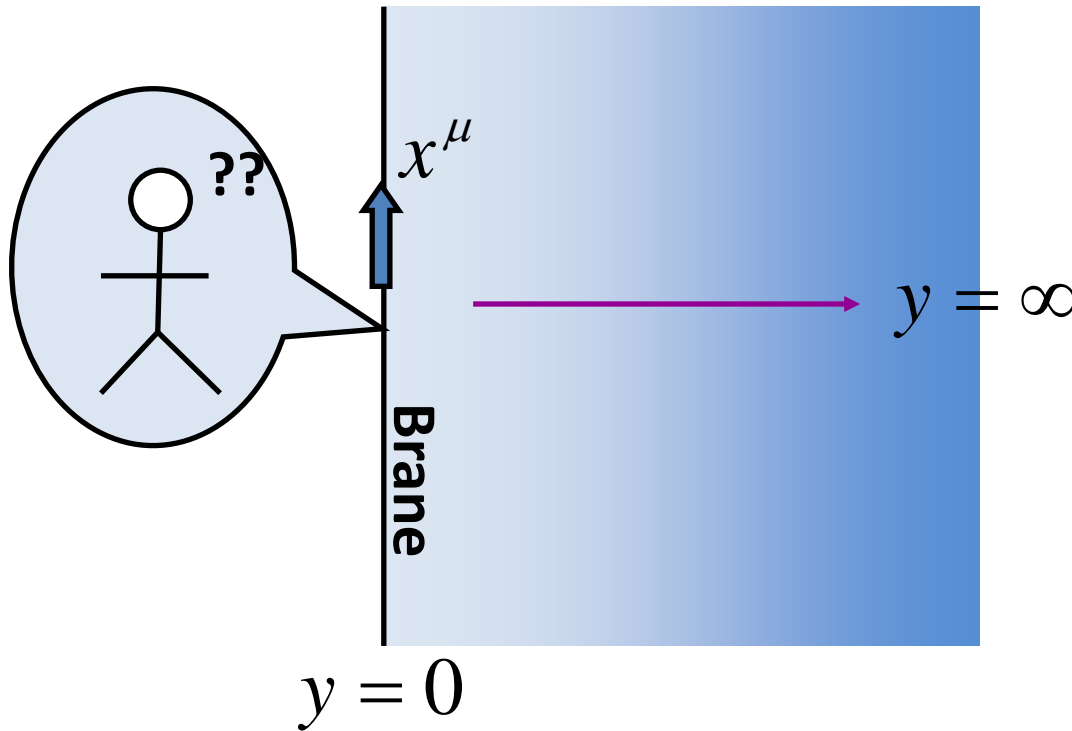
Positive tension brane

$$M_{pl}^2 = \left( 1 - e^{-2d/\ell} \right) M_5^3 \ell$$

finite for  $d \rightarrow \infty$

# Infinite braneworld model

Randall-Sundrum II model (1999)



$$M_{pl}^2 = \left(1 - e^{-2d/\ell}\right) M_5^3 \ell$$



$$M_{pl}^2 = M_5^3 \ell$$

- Infinite proper distance in the direction of extra-dimension
- compactification without compactification?

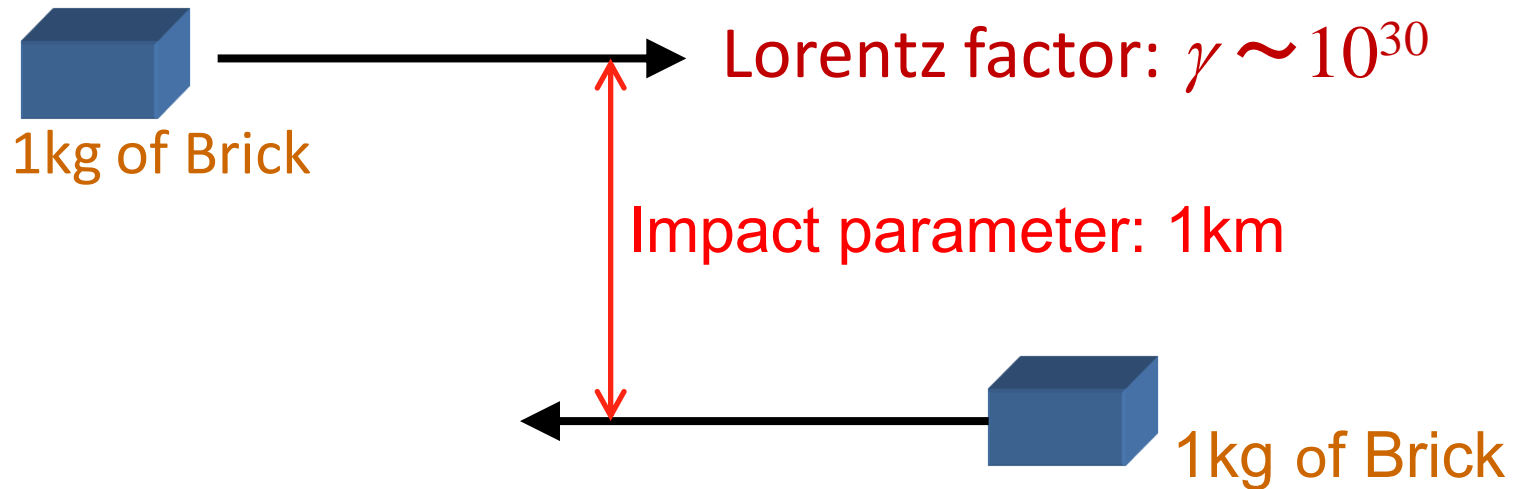
# Black hole at TeV scale


- Braneworld:
  - (fundamental scale of gravity)  $\sim$  TeV
  - ADD model
  - RS I (two branes) model
- BH formation at collider
  - usually *impossible* because

$$R_s = \frac{2M}{M_{pl}^2} > l_{pl} = \frac{1}{M_{pl}} \quad \text{requires} \quad M > M_{pl} \cdot$$

# Black hole formation?

- Situation is



The effective mass of the other seen from one brick is  $\gamma M \approx 10^{33} g$  .  Schwarzschild radius is  $\approx 1\text{km}$

Since two lumps of brick are already inside the horizon radius at the collision, a black hole may form. Otherwise, naked singularity.

# Shock wave

- Aichelburg and Sexl GRG2 303 (1971)

Boosting Schwarzschild-Tangherlini metric:

$$ds^2 = - \left( 1 - \frac{16\pi G \hat{M}}{(D-2)\Omega_{D-2}} \frac{1}{r^{D-3}} \right) dt^2 + \left( 1 - \frac{16\pi G \hat{M}}{(D-2)\Omega_{D-2}} \frac{1}{r^{D-3}} \right)^{-1} dr^2 + r^2 d\Omega_{D-2}^2$$

$$\begin{aligned} \bar{t} &= \gamma(t + \beta z) & \bar{u} &= \bar{t} - \bar{z} = \gamma(1 - \beta)(t - z) & \gamma &= \frac{1}{\sqrt{1 - \beta^2}} \\ \bar{z} &= \gamma(z + \beta t) & \bar{v} &= \bar{t} + \bar{z} = \gamma(1 + \beta)(t + z) \end{aligned}$$

$\beta \rightarrow 1$  limit is taken keeping  $\mu = \gamma \hat{M}$  fixed.

$$-dt^2 + dz^2 \rightarrow -d\bar{u}d\bar{v}$$

$$\Rightarrow ds^2 = -d\bar{u}d\bar{v} + \Phi(\bar{r})\delta(\bar{u})d\bar{u}^2 + dr^2 + \bar{r}^2 d\Omega_{D-3}^2$$

$$\bar{r}^2 = r^2 - z^2$$

$$\Phi(\bar{r})\delta(\bar{u}) = \frac{16\pi G \hat{M}}{(D-2)\Omega_{D-2}} \lim_{\gamma \rightarrow \infty} \frac{\gamma(\bar{r}^2 + 8\gamma^2 \bar{u}^2)}{(\bar{r}^2 + 4\gamma^2 \bar{u}^2)^{D-3}} = 0 \text{ except for } \bar{u} = 0$$

$$\int_{-\varepsilon}^{\varepsilon} d\bar{u} \Rightarrow \Phi(\bar{r}) = \frac{8\pi G \mu}{(D-4)\Omega_{D-3}} \frac{1}{\bar{r}^{D-4}}$$

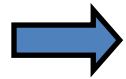
$$ds^2 = -d\bar{u}d\bar{v} + \Phi(\bar{r})\delta(\bar{u})d\bar{u}^2 + d\mathbf{x}^2$$

Further coordinate transformation to remove  $\delta$ -function:

$$\bar{u} = u$$

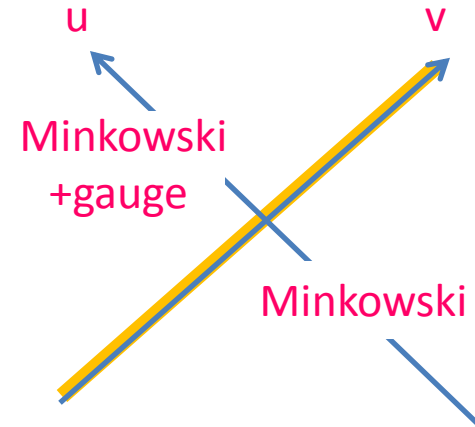
$$\bar{v} = v + \Phi\theta(u) + \frac{u\theta(u)(\nabla\Phi)^2}{4}$$

$$\bar{x}^i = x^i + \frac{u}{2}\nabla_i\Phi\theta(u)$$



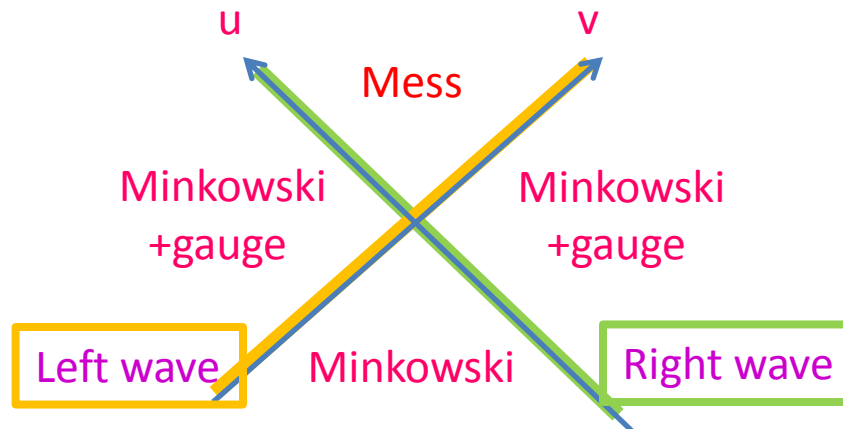
$$ds^2 = -dudv + H_{ik}H_{jk}dx^i dx^j$$

$$\text{where } H_{ij} = \delta_{ij} + \frac{1}{2}\nabla_i\nabla_j\Phi u\theta(u)$$



Taking the superposition of two shock waves:

$$ds^2 = -dudv + \left[ H_{ik}^{(R)}H_{jk}^{(R)} + H_{ik}^{(L)}H_{jk}^{(L)} - \delta_{ij} \right] dx^i dx^j$$



# Finding Apparent Horizon

- Eardley & Giddings (2002), Yoshino & Nambu (2003,2004),  $u$
- Yoshino & Rychkov (2005)

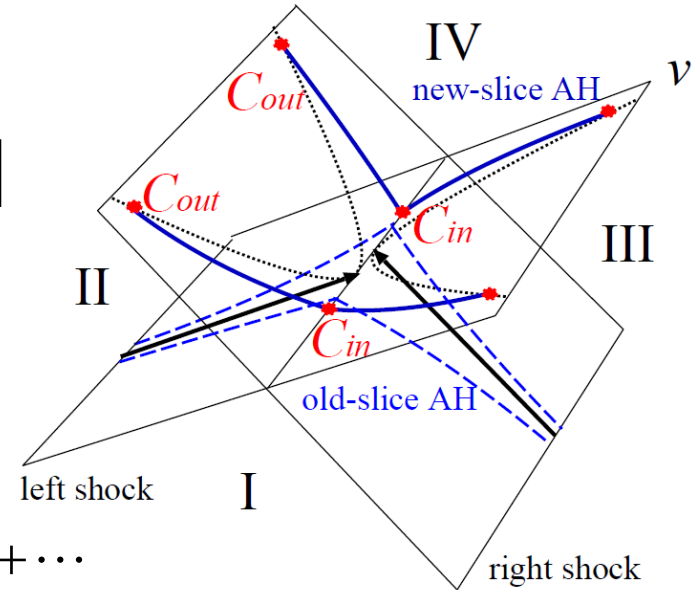
Specify the AH by two equations

$$f \equiv u - h(\rho, \phi) = 0 \quad g \equiv v = 0 \quad \rho = |\vec{x}|$$

$$\left\{ \begin{array}{l} \nabla f \text{ is normal to any tangent vectors in } f=0. \\ \nabla g \text{ is normal to any tangent vectors in } g=0. \end{array} \right.$$

➔  $k_\mu = \nabla_\mu f + s \nabla_\mu g$  is normal to AH.

$$k_\mu k^\mu = 0 \quad \text{➔} \quad s = \frac{1}{4(1 + \dots / \rho^{D-2})^2} (\partial_\rho h)^2 + \dots$$



Finding AH is basically solving a differential eq.  $\nabla_\mu k^\mu = 0$  for  $h$ .

A little complication arises to guarantee the continuity of  $k$  on the boundaries.

At  $u=0$  and  $v=0$ , the location of already AH surfaces on both sides are chosen to be identical and  $k$  is null.

If one explicit continuity condition is imposed, the others are satisfied automatically.

$$k^u / k^v |_{left} = k^v / k^u |_{right} \quad \text{➔} \quad \left( (\partial_\rho h)^2 + \rho^{-2} (\partial_\phi h)^2 \right)_{left} = \left( (\partial_\rho h)^2 + \rho^{-2} (\partial_\phi h)^2 \right)_{right} = 16$$

There also exist a coordinate singularity:

$$\begin{aligned}
 ds^2 &= -dudv + H_{ik}H_{jk}dx^i dx^j \\
 &= -dudv + \left[ 1 + (D-3)\tilde{\mu} \frac{u}{\rho^{D-2}} \theta(u) \right]^2 d\rho^2 + \rho^2 \left[ 1 - \tilde{\mu} \frac{u}{\rho^{D-2}} \theta(u) \right]^2 d\Omega_{D-3}^2
 \end{aligned}$$

We need to consider the regularity of  $k$  at the coordinate singularity at  $\rho^{D-2} = \tilde{\mu}u$ .

Since it would be too much detailed, we will skip it.

Critical impact parameter for the AH formation:  $b_{crit}$

Dimension	4	5	6	7	8	9	10	11
$b_{crit}/r_h(2\mu)$	0.84	1.24	1.47	1.59	1.66	1.72	1.76	1.79

Yoshino & Rychkov (2005)

Some calculations using numerical relativity have already been done in 4D case:

Shibata, Okawa, Yamamoto (2008),

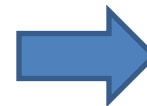
U. Sperhake, V. Cardoso, F. Pretorius, E. Berti and J.A. Gonzalez (2008)

Choptuik, Pretorius (2009)

$$b_{crit}/r_h \sim 1.25-1.5$$

$$a_{fin}/M_{fin} \sim 0.8-0.95?$$

Yoshino, Shibata (2009) 5D test calculation



Shibata-san's lecture on Thursday

# BH formation & evaporation

Formation of horizon

Cross section is  $\sigma \approx r_h^2 \approx \left( \frac{E}{M_{(n+4)}^{n+2}} \right)^{\frac{2}{n+1}}$

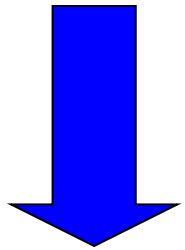
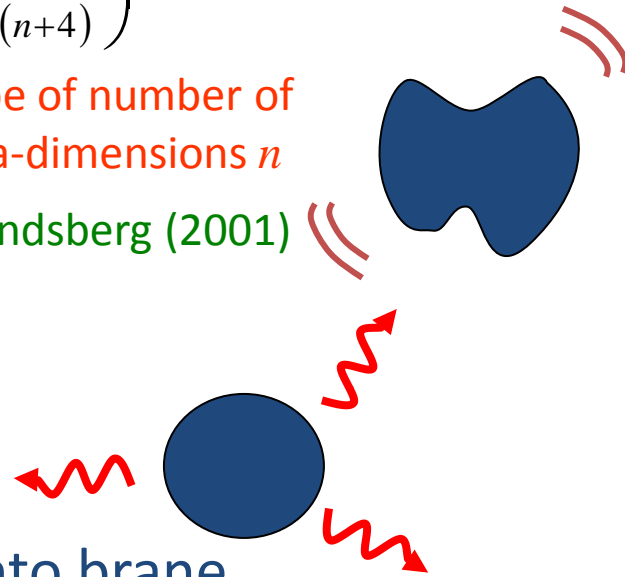
$$g_{tt}^{(Sch)} \approx 1 - \frac{E}{M_{(n+4)}^{n+2} r^{n+1}}$$

$n = D - 4$

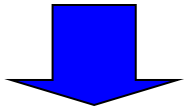
$$\log \sigma \propto \frac{2}{n+1} \log E$$

Probe of number of extra-dimensions  $n$

S. Dimopoulos, G. Landsberg (2001)



GW emission  $\Rightarrow$  Kerr BH



Hawking radiation

Most of particles are emitted into brane.

$\propto$  effective number of particle species (Emparan, Horowitz & Myers (2000))

$\Rightarrow$  detection of thermal spectrum???

But graviton emission into the bulk might be efficient because of many polarization modes.

© Escape of the BH from the brane (Flachi, Tanaka (2005))

# Quasi stationary BH

Formed black hole will settle to quasi-stationary phase.

Kerr BH (Myers-Perry BH with single rotation plane)

$$ds^2 = -\left(1 - \frac{M}{\Sigma r^{n-1}}\right) dt^2 - \frac{2aM \sin^2 \theta}{\Sigma r^{n-1}} dt d\varphi + \frac{\Sigma}{\Delta} dr^2$$

$$+ \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{a^2 M \sin^2 \theta}{\Sigma r^{n-1}}\right) \sin^2 \theta d\varphi^2 + r^2 \cos^2 \theta d\Omega_n^2$$

$$\Delta \equiv r^2 + a^2 - \frac{M}{r^{n-1}} \quad \Sigma \equiv r^2 + a^2 \cos^2 \theta \quad n = D - 4$$

More generic Myers-Perry BH solutions:

Odd D)

$$ds^2 = -dt^2 + \sum_{i=1}^{(D-1)/2} (r^2 + a_i^2) (d\mu_i^2 + \mu_i^2 d\phi_i^2) + \frac{\mu r^2}{\pi F} \left(dt - \sum_i a_i \mu_i^2 d\phi_i\right)^2 + \frac{\Pi F}{\Pi - \mu r^2} dr^2$$

Even D)

$$ds^2 = -dt^2 + r^2 d\alpha^2 + \sum_{i=1}^{D/2-1} (r^2 + a_i^2) (d\mu_i^2 + \mu_i^2 d\phi_i^2) + \frac{\mu r}{\pi F} \left(dt - \sum_i a_i \mu_i^2 d\phi_i\right)^2 + \frac{\Pi F}{\Pi - \mu r} dr^2$$

$$F = 1 - \sum_i \frac{a_i^2 \mu_i^2}{r^2 + a_i^2} \quad \Pi = \prod_i (r^2 + a_i^2)$$

# Hawking radiation from quasi stationary BH

## Thermal radiation from BH:

$$T_H = \frac{\kappa}{2\pi} = \frac{1}{2\pi} \frac{\partial \sqrt{g^{tt}}}{\sqrt{g_{rr}} \partial r} \quad : \text{BH temperature} = \text{surface gravity}$$

$$ds^2 \xrightarrow{\text{near horizon}} -\rho^2 (2\pi T_H dt)^2 + d\rho^2 + (d\phi - \underline{\Omega_H} dt)^2 + \dots$$

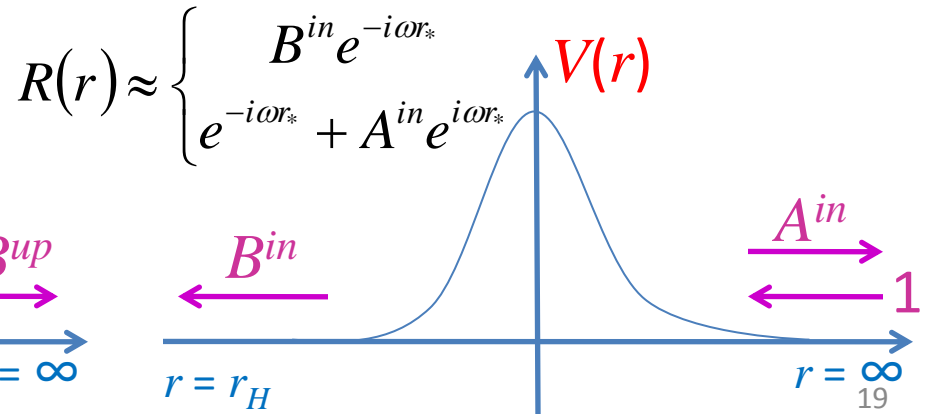
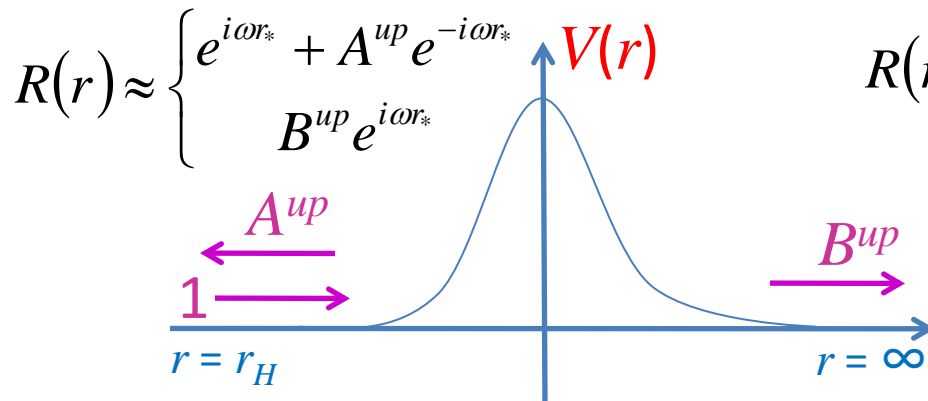
BH rotation velocity

Hawking radiation is basically thermal radiation, but transmission rate between the horizon and infinity should be taken into account.

$$\frac{dN(\omega)}{dt d\omega} \equiv \frac{1}{2\pi} \frac{|\zeta(\omega)|^2}{\exp(\tilde{\omega}/T_H) \mp 1}$$

$$\tilde{\omega} = \omega + m\Omega_H$$

$$|\zeta(\omega)|^2 = |B^{up}|^2 = |B^{in}|^2 = 1 - |A^{in}|^2$$



# Hawking radiation from higher dimensional BHs

Non-rotating BH case

taken from [Cardoso, Cavaglia, Gualtieri, JHEP 0602:021 \(2006\)](#).

Fraction of emission rates per d.o.f. normalized to the scalar field

$D$	4	5	6	7	8	9	10	11
Scalars	1	1	1	1	1	1	1	1
Fermions	0.37	0.7	0.77	0.78	0.76	0.74	0.73	0.71
Gauge Bosons	0.11	0.45	0.69	0.83	0.91	0.96	0.99	1.01
Gravitons	0.02	0.2	0.6	0.91	1.9	2.5	5.1	7.6

Percentage of power going into each field species.

$D$	4	5	6	7	8	9	10	11
Scalars	6.8	4.0	3.7	3.6	3.6	3.5	3.3	2.9
Fermions	83.8	78.7	75.0	72.3	69.9	66.6	61.6	53.4
Gauge Bosons	9.3	16.7	20.0	21.7	22.3	22.2	20.7	18.6
Gravitons	0.1	0.6	1.3	2.4	4.2	7.7	14.4	25.1

Number of higher dimensional gravitons : 
$$\frac{D(D-1)}{2} - 2 - (D-2) = \frac{D(D-3)}{2}$$

## Rotating BH case:

For brane localized fields, we can use the well-known four dimensional standard technique (Petrov type-D)

= Newman-Penrose-Teukolsky formalism

(e.g., Ida, Oda & Park)

For bulk fields, the perturbation equations are derived only for scalar field (Frolov, Krtouš and Kubizňák (2007)) and for Fermion Field (Oota & Yasui (2008)).

Bulk graviton calculation is limited to restricted cases;

Odd-dimensional, equal angular momentum, tensor

(Kunduri, Lucietti & Reall(2006))

5dim, equal angular momentum (Murata & Soda (2008))

$D \geq 7$ , tensor type (Kodama, Konoplya & Zhidenko (2009))

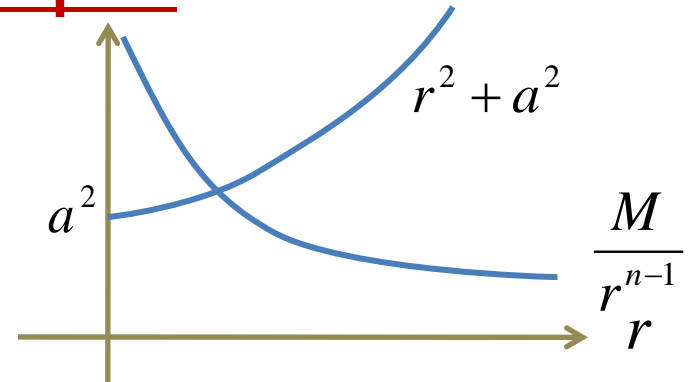
# Maximum spin

Kerr parameter  $a$  does not have bound for  $n \geq 2$ .

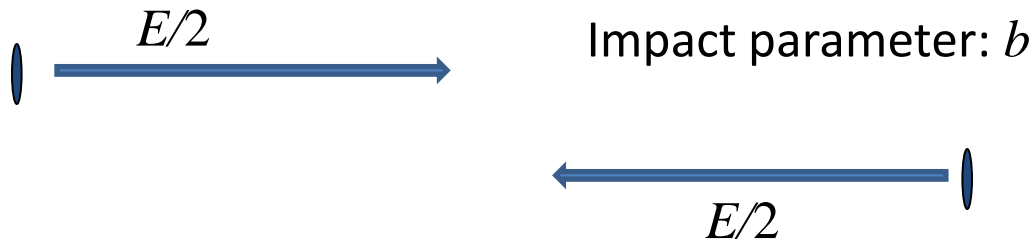
$$\Delta \equiv r^2 + a^2 - \frac{M}{r^{n-1}}$$

For  $n=1$ ,  $a_{\max} = M^{1/2}$

For  $n \geq 2$ ,  $a_{\max} = \infty$



However, formation of really ultra-spinning BH seems to be unlikely.



Radius of the formed black hole should be larger than the impact parameter:

$$b < r_h$$

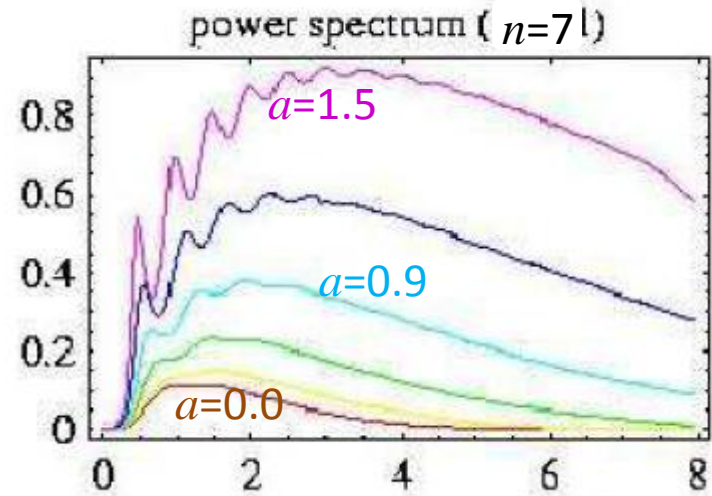
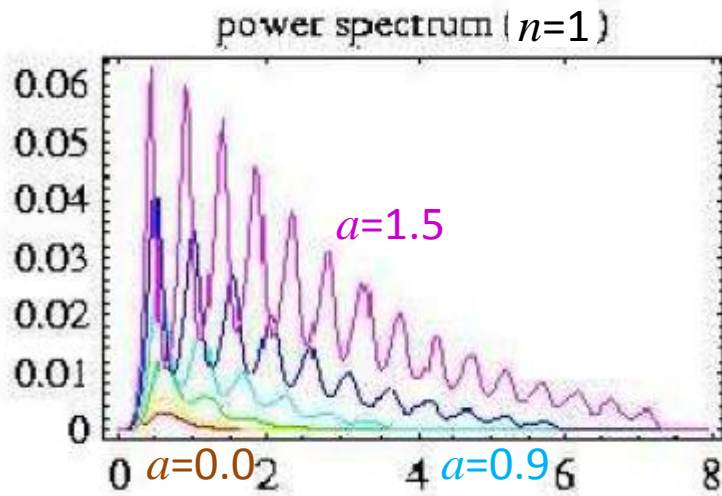
Angular momentum:  $J \equiv \frac{2aE}{n+2} \approx 2(E/2)b$

$$\begin{aligned} \rightarrow \frac{2a_{\max}}{n+2} \approx r_h & \quad \rightarrow a_{\max} / M^{1/(n+1)} = \begin{cases} 1.25 & (n=2) \\ 1.89 & (n=3) \\ 2.46 & (n=4) \end{cases} \end{aligned}$$

# Summary of Lecture I

- Black hole might be produced at collider in higher dimensional spacetime modes.
- Formation rate of BH can be rough estimated by using the two boosted BHs, but reliable calculation should be based on numerical relativity.
- Emitted particles via Hawking evaporation will mainly go to brane particles, but bulk graviton emission rate is still difficult to study for rotating BHs.
- Highly rotating BHs have peculiar feature in its Hawking radiation. But its significance depends on the maximum rotation, which is poorly understood.  
⇒ continues to Lecture II

# Hawking radiation looks thermal?

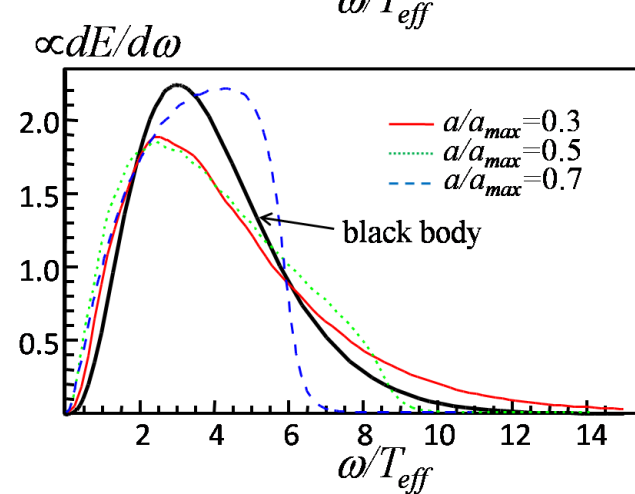
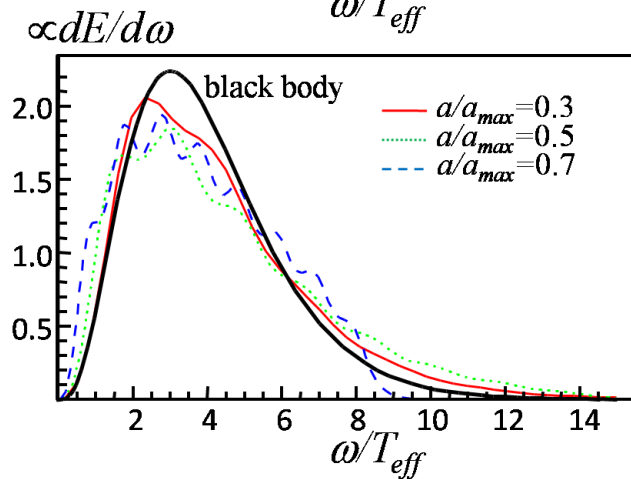
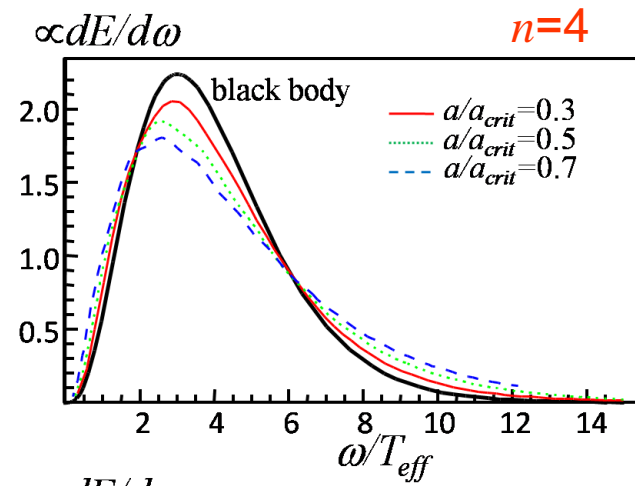
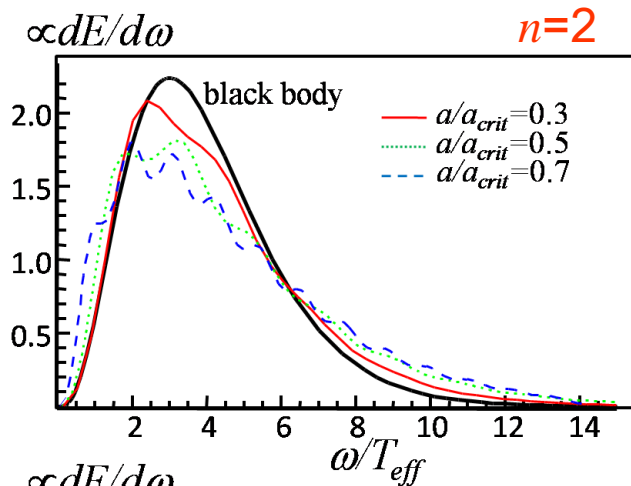


vector particles

(Ida, Oda & Park (2006))

Is deviation from the thermal spectrum clear?

# Hawking radiation looks almost thermal



In normalized plots using  $T_{eff} (\neq T_H)$ ,  
deviation from the thermal spectrum is not so clear.

# Superposition of thermal spectra

The numerical data can be fit well by an appropriate superposition of thermal spectrum.

$$\Delta T \approx 2\Omega_h \left( M / M_{n+4}^{n+2} \right)^{1/(n+1)} \times T_{eff}$$

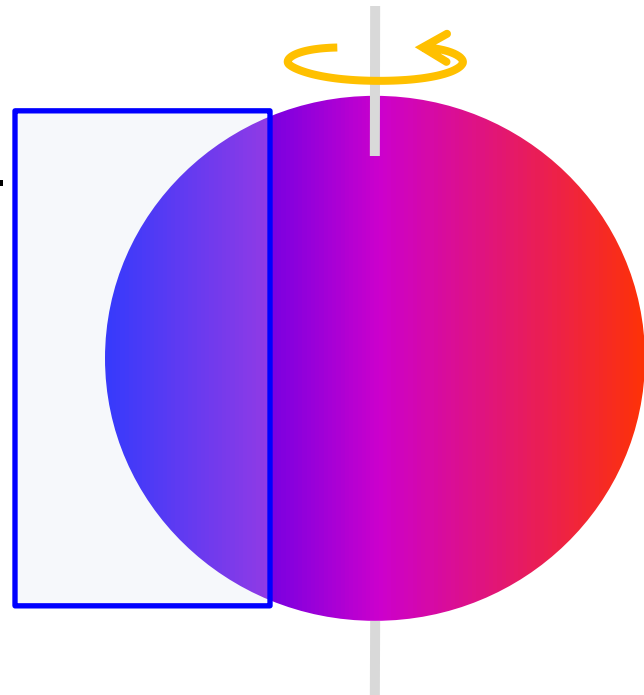
However, through the evaporation process,  $T$  and  $a$  change gradually. So, anyway what we observe is a superposition of different  $T$ .

➔ It will be also difficult to detect wiggles.

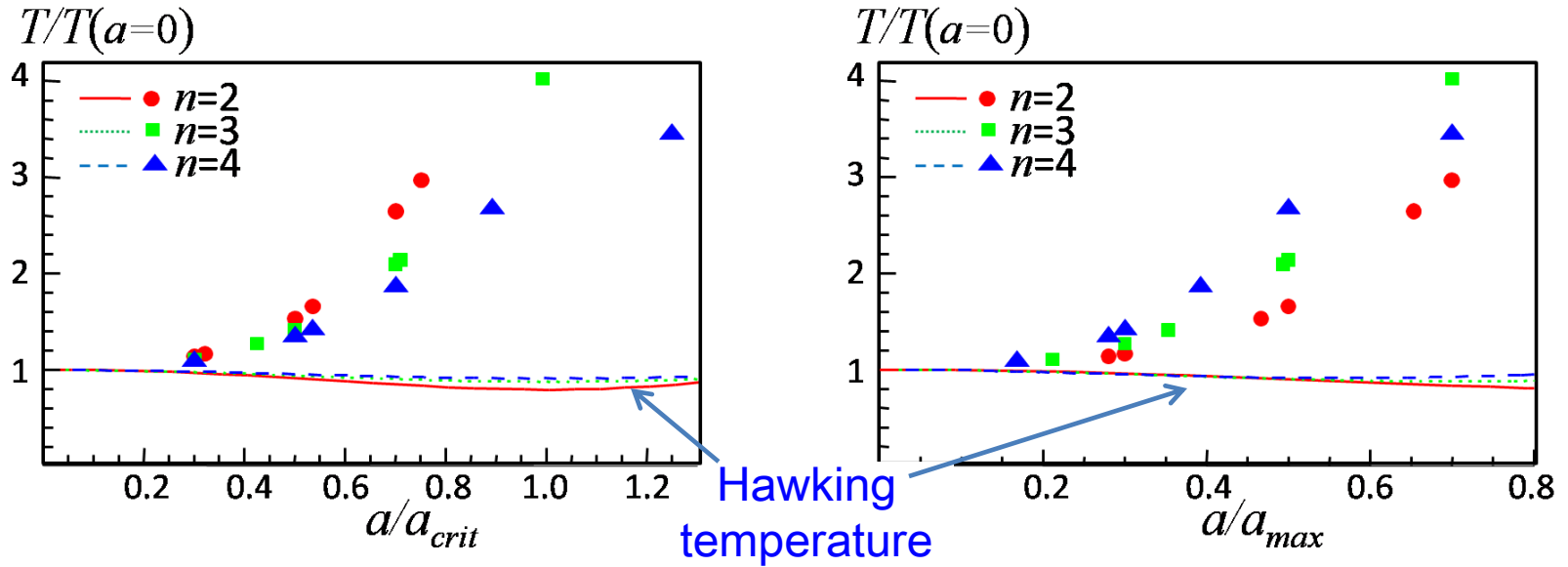
Why superposition?

Imagine a rotating black body sphere.

This part dominates radiation spectrum.



# Effective temperature of Hawking radiation



For larger  $n$ , effective temperature becomes low or high?

Answer depends how we fix  $a$ .

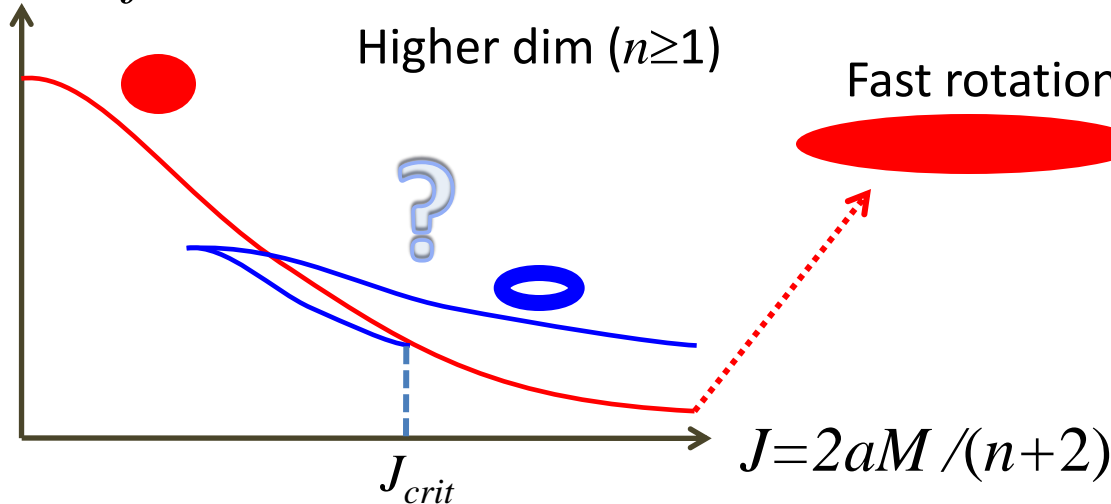
If we can determine both  $a$  and  $T$ , we might be able to discriminate different # of extra-dimensions,  $n$ .

c.f., We also discussed the use of spin-dependent distribution in **JHEP 0905:031,2009**

# Is there a stability changing point?

(Empanan, Myers (2003))

Area of horizon



Higher dim ( $n \geq 1$ )

Fast rotation limit

Geometry near the axis looks like black  $p$ -brane.

Gregory-Laflamme instability

$$n = D - 4$$

Sectional metric at  $t = \text{constant } r = r_H$ :

$$ds^2 = \Sigma d\theta^2 + \frac{(r_H^2 + a^2)^2}{\Sigma} \sin^2 \theta d\phi^2 + r^2 \cos^2 \theta d\Omega_n^2$$

$$\Delta(r_H) = r_H^2 + a^2 - \frac{M}{r_H^{n-1}} = 0 \quad a \rightarrow \infty \implies r_H \rightarrow (M / a^2)^{1/(n-1)}$$

$\theta, \phi$ -directions)  $A^{(2)} = (r_H^2 + a^2) \Omega_2 \rightarrow a^2 \Omega_2$

the other directions)  $A^{(n)} = (r_H \cos \theta)^n \Omega_n$



Large horizon in two-dimensions for large  $a$

$$ds^2 = -\left(1 - \frac{M}{\Sigma r^{n-1}}\right) dt^2 - \frac{2aM \sin^2 \theta}{\Sigma r^{n-1}} dt d\varphi + \frac{\Sigma}{\Delta} dr^2$$

$$+ \Sigma d\theta^2 + \left(r^2 + a^2 + \frac{a^2 M \sin^2 \theta}{\Sigma r^{n-1}}\right) \sin^2 \theta d\varphi^2 + r^2 \cos^2 \theta d\Omega_n^2$$

Keeping  $\sigma = a \sin \theta$  and  $\mu = M / a^2$  to be finite, take the limit  $a \rightarrow \infty$ .

$$\Sigma = r^2 + a^2 \cos^2 \theta \rightarrow a^2$$

$$\Delta = r^2 + a^2 - \frac{M}{r^{n-1}} \rightarrow a^2 \left(1 - \frac{\mu}{r^{n-1}}\right)$$

$$\Rightarrow ds^2 \rightarrow -\left(1 - \frac{\mu}{r^{n-1}}\right) dt^2 + \left(1 - \frac{\mu}{r^{n-1}}\right)^{-1} dr^2 + r^2 d\Omega_n^2 + \frac{d\sigma^2 + \sigma^2 d\varphi^2}{R^2}$$

$BH \otimes \mathbb{R}^2$ : black brane solution.

## Gregory-Laflamme instability:

Black  $p$ -brane is unstable when the extension in the direction of the translational symmetry is larger than the radius of horizon in the other directions.

Therefore highly spinning black hole is expected to be unstable.

# Where is the onset of instability?

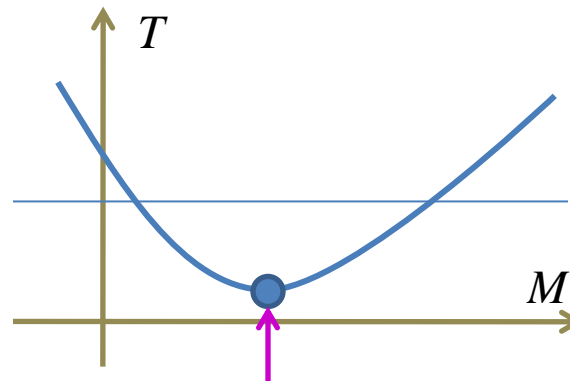
“Gregory-Laflamme instability point

= Thermodynamical instability point in canonical ensemble”

when there is an exact spatial translational symmetry (Gubser-Mitra conjecture)

Spherically  
symmetric case:  
Ex.) Sch-AdS Black String

$$dS = T^{-1} dM$$



For the same T,  
there are two sols.

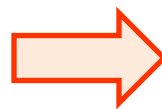


One is stable and  
the other is not.

$dT/dM=0$  at the stability changing point

$$dS = T^{-1} (dM - \Omega_H dJ)$$

$$\det \begin{pmatrix} \partial T / \partial M & \partial T / \partial J \\ \partial \Omega_h / \partial M & \partial \Omega_h / \partial J \end{pmatrix} = 0$$



$$a_{crit}^2 = \frac{n+1}{n-1} r_h^2$$

$$a_{crit} / M^{1/(n+1)} = \begin{cases} 1.09 & (n=2) \\ 1.07 & (n=3) \\ 1.06 & (n=4) \end{cases}$$

# Stability analysis

It will be important at which value of  $a$  Myers-Perry BH becomes unstable.

If instability occurs via a kind of GL-instability, the longest wavelength mode is the most unstable.

We consider  $U(1) \times O(n)$ -sym modes. (most symmetric)

$$h \approx e^{-i\omega t + im\phi} H(r, \theta)$$

$\phi$ -rotation

In this  $m=0$  case we can write the perturbation equation formally as

$$\Delta \psi_{\mu\nu} = -\omega^2 \psi_{\mu\nu}$$

Hence, the eigen value  $\omega^2$  will be real.

Onset of instability  $\longleftrightarrow$  Stationary perturbation  
 $\omega^2 = 0$  mode appears

# Equation for stationary configuration

$$\delta G_{\mu\nu} = 0 \implies \left\{ \Delta_{(2)} H_i(r, \theta) \right\} = 0 \quad \text{Two-dimensional Laplace-type eqs.}$$

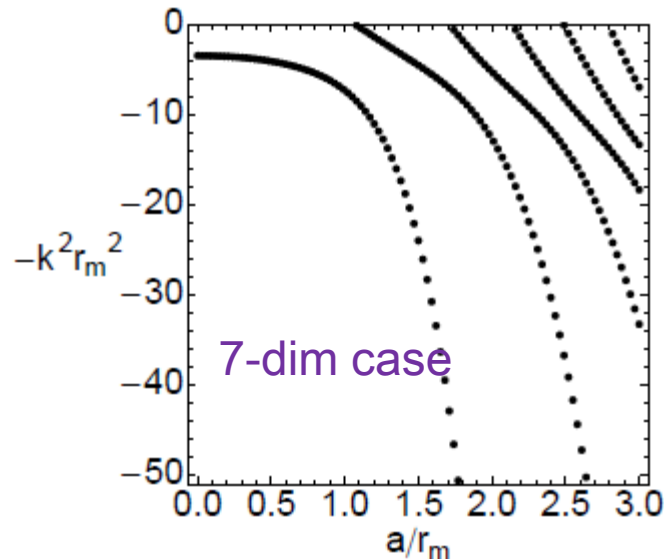
Solution exists only for a specific value of  $a$ .

Instead, we consider a relaxed problem.

$$\left\{ \Delta_{(2)} H_i \right\} = \lambda H_i \quad \text{eigen-value problem.}$$

$$\lambda = \lambda_1, \lambda_2, \lambda_3, \dots$$

When some eigenvalue  $\lambda$  crosses 0, there exists a stationary solution.



by Dias et al.

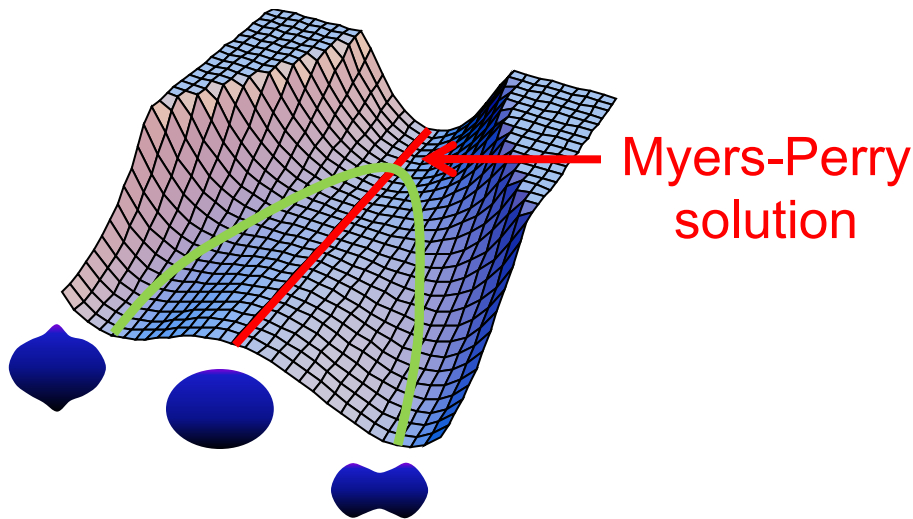
# New sequence of solutions

Onset of GL-type instability

Dimension	6	7	8	9
$a_{inst}/r_h$	****	3.09	2.70	2.49
$a_{inst}/M^{1/(n-1)}$	****	1.714	1.770	1.792

Dias, Figueras, Monteiro, Santos, & Emparan (2009)

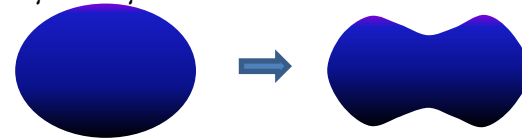
Onset of instability means appearance of new sequences of solutions.



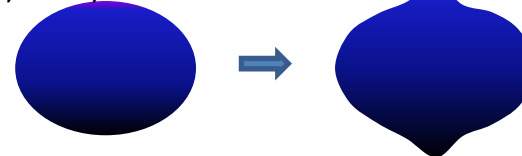
Myers-Perry solution

New stable solutions?

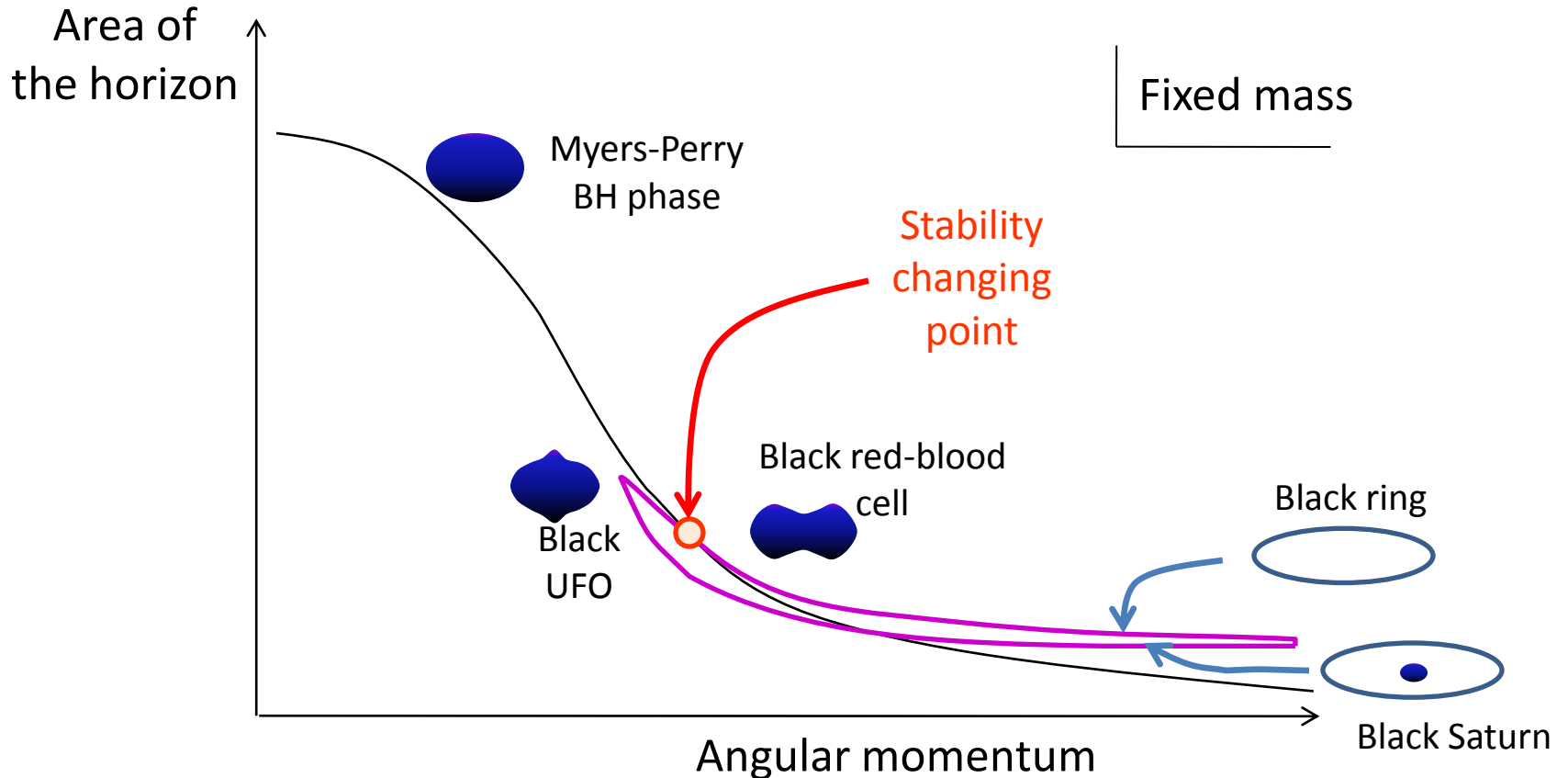
If  $g_{\mu\nu} + h_{\mu\nu}$  is a stationary perturbation,



$g_{\mu\nu} - h_{\mu\nu}$  is also a stationary perturbation



# Expected phase structure



Even if non-axisymmetric instability may exist for smaller angular momentum, non-trivial phase structure of sequences of stationary solutions would be interesting. ➡ Shibata-san's lecture  
In 5-dimensional case, exact black ring/Saturn solutions exist.

# 5dimensional Black Ring

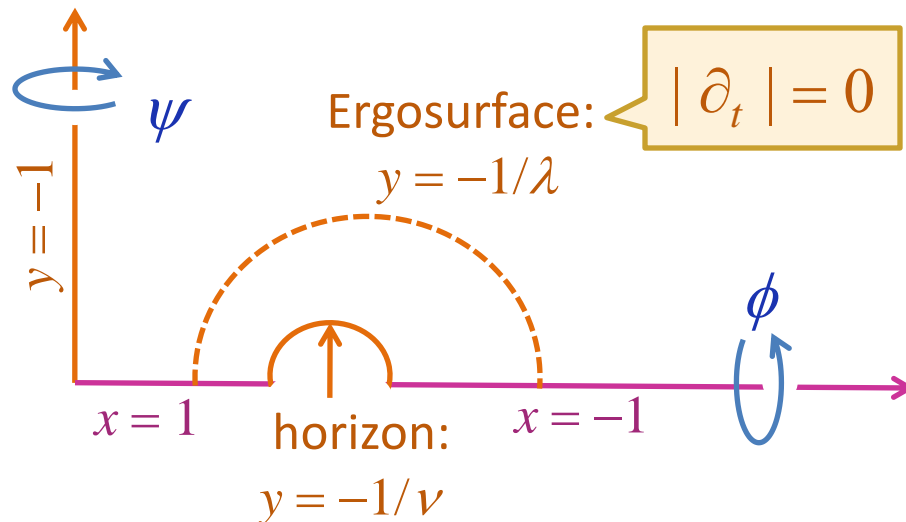
Black ring exact solution:

(Empanan, Reall (2002))

$$ds^2 = -\frac{F(y)}{F(x)} \left( dt - CR \frac{1+y}{F(y)} d\psi \right)^2 + \frac{R^2}{(x-y)^2} F(x) \left( -\frac{G(y)}{F(y)} d\psi^2 - \frac{dy^2}{G(y)} + \frac{dx^2}{G(x)} + \frac{G(x)}{F(x)} d\phi^2 \right)$$

$$F(\xi) \equiv 1 + \lambda\xi \quad G(\xi) \equiv (1 - \xi^2)(1 + \nu\xi) \quad \lambda \equiv \frac{2\nu}{1 + \nu^2} \quad C \equiv \sqrt{\lambda(\lambda - \nu) \frac{1 + \lambda}{1 - \lambda}}$$

$$-\infty \leq y \leq -1 \quad -1 \leq x \leq 1 \quad 0 \leq \nu \leq 1$$



Metric looks singular,  
but it is not.

$R$  : radius  
 $\nu$  : thickness

# Solutions with (D-2) commuting Killing vectors

$$ds^2 = g_{ab}(r, z)dx^a dx^b + e^{2\nu(r, z)}(dr^2 + dz^2)$$

In 5D, 3 commuting Killing vectors  $\partial_t, \partial_{\phi_1}, \partial_{\phi_2}$  are allowed.

In 6D, 4 commuting Killing vectors are not compatible with asymptotic flatness.

**Einstein eqs)**  $\partial_r U + \partial_z V = 0$

$$U \equiv r(\partial_r g)g^{-1} \quad V \equiv r(\partial_z g)g^{-1}$$

$$\partial_r \nu \equiv -\frac{1}{2r} + \frac{1}{8r} \text{Tr}(U^2 - V^2) \quad \partial_z \nu \equiv \frac{1}{4r} \text{Tr}(UV)$$

**Belinsky Zakharov transformation:** (Belinski & Verdaguer, “Gravitational solitons”

Without solving differential eqs., Emparan & Reall, Liv. Rev. Rel **11**, 6 )

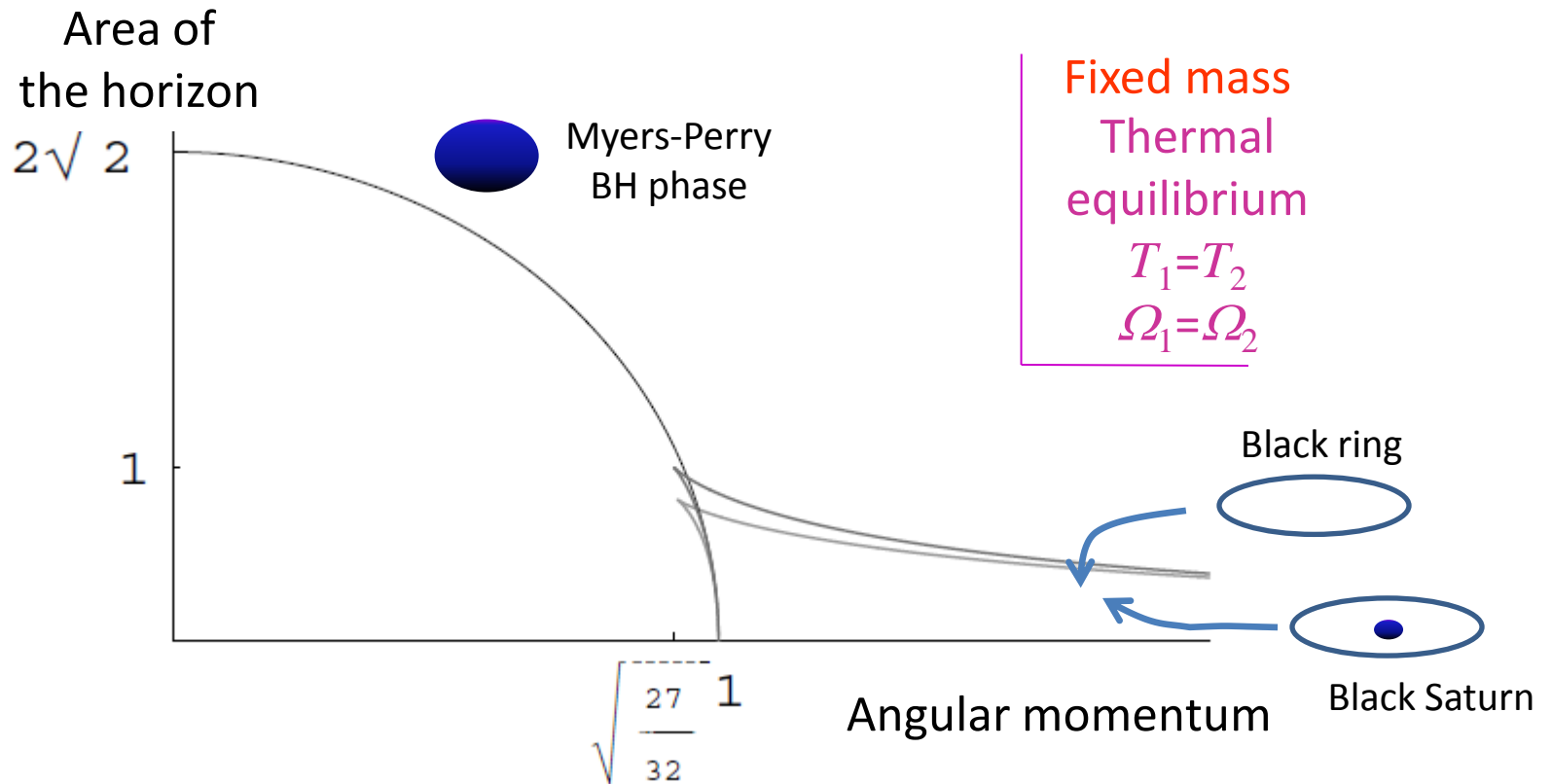
one can generate new solutions.

Black Saturn, (Elvang & Figueras (2007))

Concentric Black di-ring, (Iguchi & Mishima (2007), Evslin and C. Krishnan (2007))

Orthogonal Black di-ring, (Izumi (2008); Elvang & Rodriguez(2008))

# Phase structure of 5D black objects



Even in higher dimensional cases, the limit of large angular momentum can be studied rather easily.

# Blackfolds

In higher dimensions exact black ring solutions are not found.  
 But an approximate solution can be constructed for “length” > “width”,  
 although such black objects are unstable (GL instability).

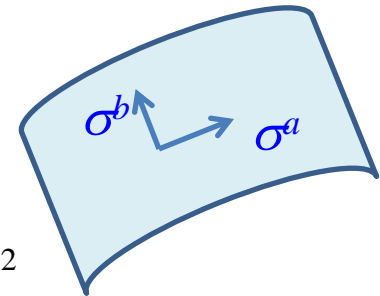
Matched asymptotic expansion :

$$n = D - 3 - p$$

Near zone metric: boosted black  $p$ -brane

$$ds^2 \approx -\left(1 - \frac{r_0^n}{r^n}\right) dt^2 + \sum_{i=1}^p (dz^i)^2 + \left(1 + \frac{r_0^n}{r^n}\right) d\mathbf{x}_{n+2}^2$$

boost  $\rightarrow ds^2 \approx \left(\gamma_{ab} - \frac{r_0^n}{r^n} u_a u_b\right) d\sigma^a d\sigma^b + \left(1 + \frac{r_0^n}{r^n}\right) d\mathbf{x}_{n+2}^2$



$\Sigma$ :  $p$ -dimensional surface specified by  $X^\mu(\sigma^a)$

Far zone metric: background + perturbation

$$\square h_{\mu\nu} = -16\pi G \left( T_{\mu\nu} - \frac{1}{D-2} \eta_{\mu\nu} T \right)$$

with  $T_{\mu\nu} = \underline{(\varepsilon + P)u^\mu u^\nu + Ph^{\mu\nu}} + \dots$  higher order terms depending on curvature  
 Perfect fluid form

$$h^{\mu\nu} = \gamma^{ab} \partial_a X^\mu \partial_b X^\nu \quad \varepsilon = \frac{\Omega_{n+1}}{16\pi G} (n+1) (r_0(\sigma^a))^n \quad P = -\frac{1}{n+1} \varepsilon$$

$$\perp_{\rho}^{\nu} T_{\mu\nu} = 0 \text{ and the conservation law } \Rightarrow h_{\mu}^{\nu} \nabla_{\nu} T^{\mu\rho} = 0$$

$$\perp_{\mu\nu} \equiv g_{\mu\nu} - h_{\mu\nu}$$

Contracted  
extrinsic  
curvature

**Substituting**  $T_{\mu\nu} = (\varepsilon + P)u^{\mu}u^{\nu} + Ph^{\mu\nu} + \dots$

$$\Rightarrow u^{\mu}u^{\nu} \nabla_{\nu} \varepsilon + (\varepsilon + P)(\dot{u}^{\mu} + u^{\mu} \nabla_{\nu} u^{\nu}) + (h^{\mu\nu} + u^{\mu}u^{\nu}) \nabla_{\nu} P + P \nabla_{\nu} h^{\mu\nu} = 0$$

$\equiv h_{\nu}^{\rho} \nabla_{\rho} \varepsilon$ 
 $\equiv K^{\mu}$  :

Substituting  
 $\varepsilon$  and  $P$

Projection in  $u$ -direction:

$$u^{\nu} \nabla_{\nu} \varepsilon + (\varepsilon + P) \nabla_{\nu} u^{\nu} = 0$$

In the other direction:

$$(\varepsilon + P) \dot{u}^{\mu} = -(h^{\mu\nu} + u^{\mu}u^{\nu}) \nabla_{\nu} P - PK^{\mu}$$

$$\dot{u}^{\mu} + \frac{1}{n+1} u^{\mu} \nabla_{\nu} u^{\nu} = \frac{1}{n} K^{\mu} + \bar{\nabla}^{\mu} \ln r_0$$

1) projection parallel to the brane:

$$h_{\mu\nu} \dot{u}^{\nu} + \frac{1}{n+1} u_{\mu} \nabla_{\nu} u^{\nu} = \bar{\nabla}_{\mu} \ln r_0$$

2) projection normal to the brane:  $n \perp_{\mu}^{\rho} \dot{u}^{\mu} = K^{\rho}$

1) parallel to the brane

$$h_{\mu\nu}\dot{u}^\nu + \frac{1}{n+1}u_\mu\bar{\nabla}_\nu u^\nu = \bar{\nabla}_\mu \ln r_0$$

2) projection normal to the brane

$$n \perp^\rho_\mu \dot{u}^\mu = K^\mu$$

Stationary case:  $u^\mu = k^\mu / |k|$  on  $\Sigma$  with  $k^\mu$  being a Killing vector

$$k^\mu k^\nu \nabla_{(\mu} k_{\nu)} = 0 \quad \Rightarrow \quad k^\mu \partial_\mu |k| = 0 \quad \nabla_{(\mu} k_{\nu)} = 0$$

$$\dot{u}^\mu = \frac{k^\nu}{|k|} \nabla_\nu \frac{k^\mu}{|k|} = -\frac{k^\nu}{|k|^2} \nabla^\mu k_\nu = \nabla^\mu \ln |k|$$

1)  $\bar{\nabla}_\nu u^\nu = 0 \Rightarrow \bar{\nabla}_\nu \ln |k| = \bar{\nabla}_\nu \ln r_0$  Ring radius changes in proportion to the redshift factor.  
 $\Rightarrow r_0 \propto |k|$

2) For black ring in flat background:

$$k^\mu = (\partial_t)^\mu + \Omega(\partial_\phi)^\mu$$

Force balance

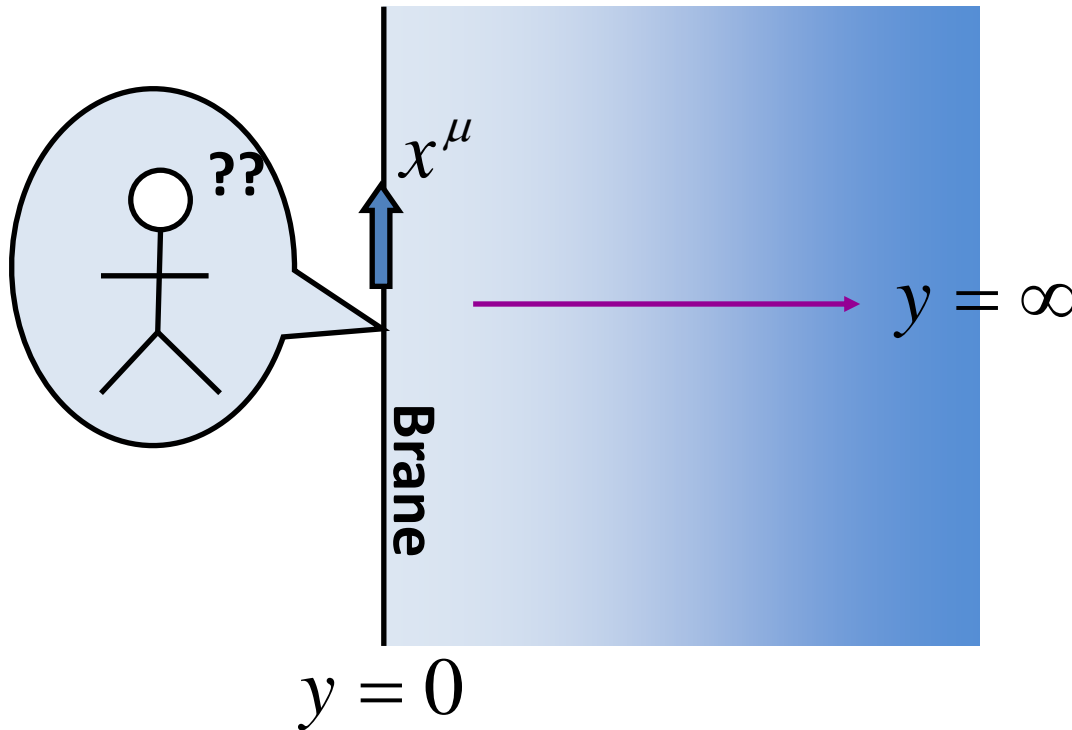
$$\Rightarrow |k| = \sqrt{1 - \Omega^2 R^2} \Rightarrow \dot{u}|_R = \frac{\Omega^2 R}{1 - \Omega^2 R^2} = \frac{1}{nR}$$

between centrifugal force and tension

In this way existence of ring solution is predicted.

# Infinite braneworld model

Randall-Sundrum II model (1999)



$$M_{pl}^2 = \left(1 - e^{-2d/\ell}\right) M_5^3 \ell$$



$$M_{pl}^2 = M_5^3 \ell$$

- Extension in the direction of extra-dimension is infinite, but 4D gravity is approximately reproduced.
- ◆ However, there is no Schwarzschild-like BH solution.

# Black string solution

( Chamblin, Hawking, Reall ('00) )

$$ds^2 = \frac{\ell^2}{z^2} \left( dz^2 + \bar{g}_{\mu\nu}^{(Sch)} dx^\mu dx^\nu \right)$$

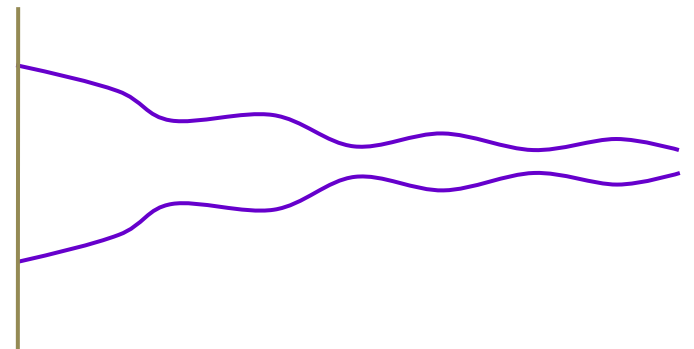
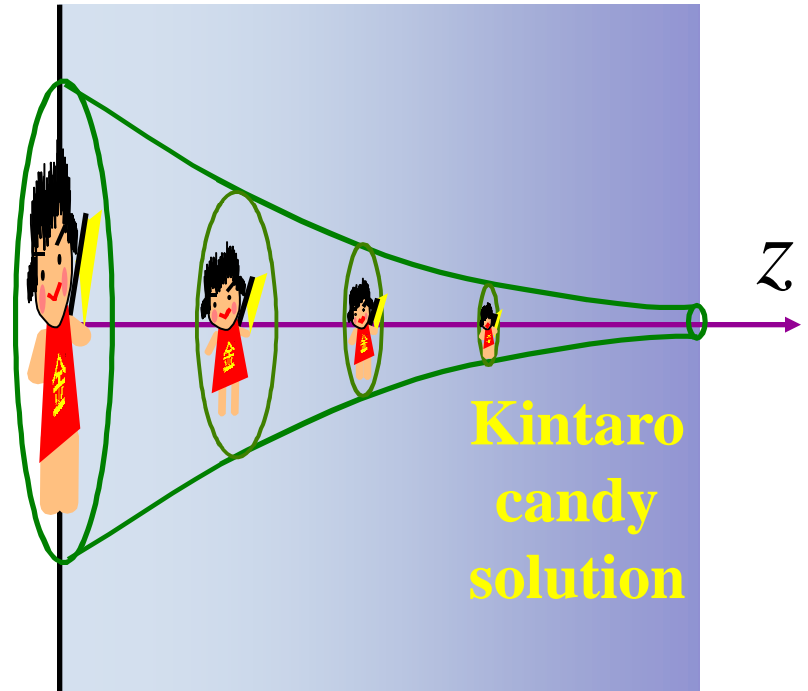
Metric induced on the brane  $\bar{g}_{\mu\nu}(x)$   
is exactly Schwarzschild solution.

However, this solution is singular.

- $C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \propto z^4$   
*behavior of zero mode*

Moreover, this solution is unstable.

- *Gregory Laflamme instability*  
“length  $\gtrsim$  width”



# AdS/CFT correspondence

( Maldacena ('98) )

( Gubser ('01) )

( Hawking, Hertog, Reall ('00) )

$$\# \underbrace{Z[q]} = \int d[\phi] \exp(-S_{CFT}[\phi, q])$$

$$\text{Boundary metric} = \int d[g_{bulk}] \exp(-S_{EH} - S_{GH} + \underbrace{S_1 + S_2 + S_3}) \equiv \exp(-W_{CFT}[q])$$

$$S_{EH} = -\frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left( {}^{(5)}R + \frac{12}{\ell^2} \right)$$

$$S_{GH} = -\frac{1}{\kappa_5^2} \int d^4x \sqrt{-q} K$$

Counter terms

$$S_1 = -\frac{3}{\kappa_5^2 \ell} \int d^4x \sqrt{-q}$$

$$S_2 = -\frac{\ell}{4\kappa_5^2} \int d^4x \sqrt{-q} {}^{(4)}R$$

$$S_3 = \dots$$

$\# \underbrace{z_0} \rightarrow 0$  limit is well defined with the counter terms.

Brane position  $z_0 \Leftrightarrow$  cutoff scale parameter

$$\# \int d[g] \exp(-S_{RS}) = \int d[g] \exp(-2(S_{EH} + S_{GH}) + 2S_1 - S_{matter})$$

$$= \exp(-2S_2 - S_{matter} - 2(W_{CFT} + S_3))$$

4D Einstein-Hilbert action

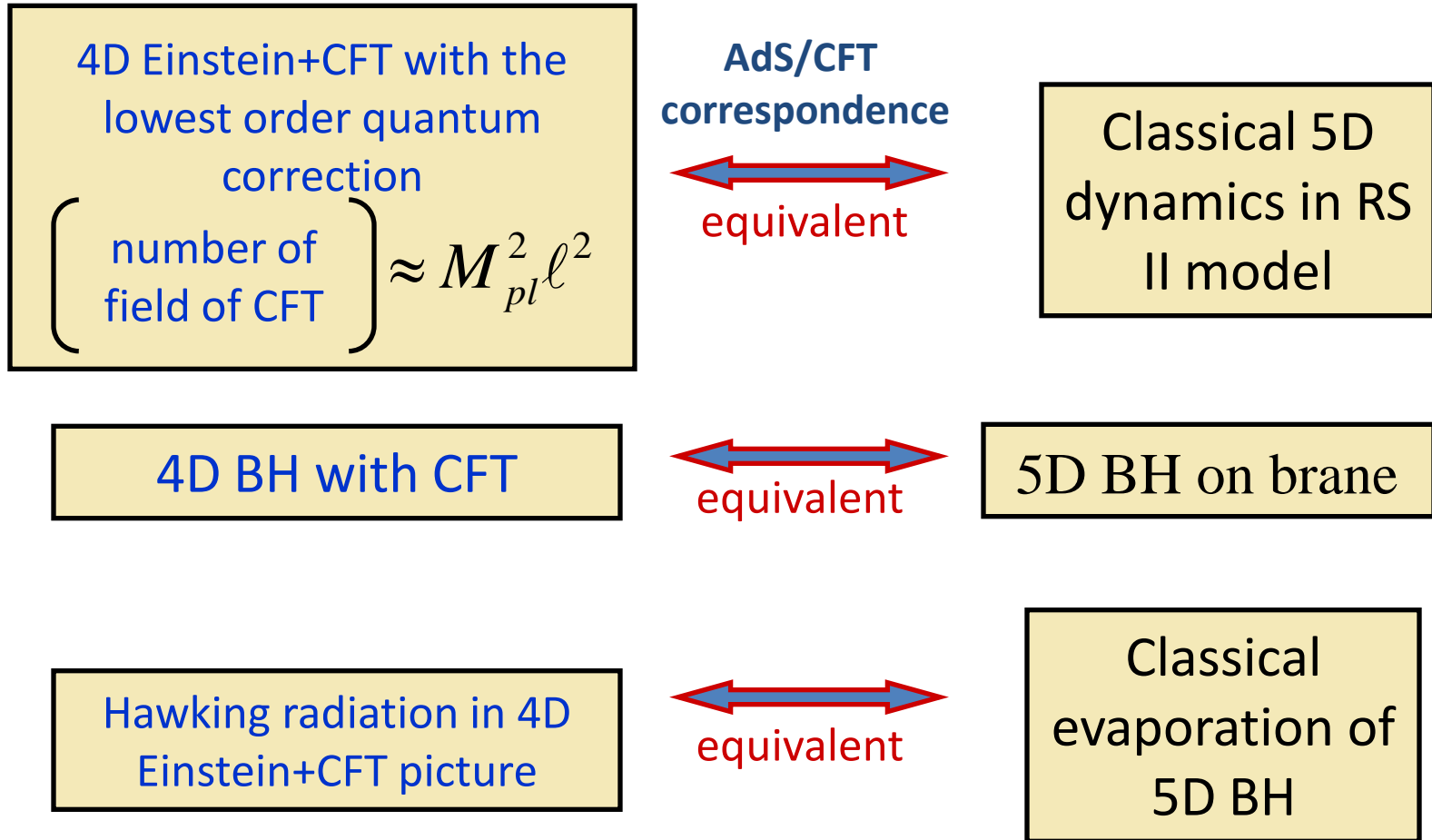
brane tension

# Evidences for AdS/CFT correspondence

- + Linear perturbation around flat background (Duff & Liu ('00))
- + Friedmann cosmology ( Shiromizu & Ida ('01) )
- + Localized Black hole solution in 3+1 dimensions  
( Emparan, Horowitz, Myers ('00) )
- + Tensor perturbation around Friedmann ( Tanaka )

# Classical black hole evaporation conjecture

(T.T. ('02), Emparan et al ('02))



## Time scale of BH evaporation

$$\tau = \left( \frac{M}{M_{Solar}} \right)^3 \left( \frac{1\text{mm}}{\ell} \right)^2 \times 120\text{year}$$

$$\frac{\dot{M}}{M} \approx \left( \text{Number of species} \right) \times \frac{1}{G_N^2 M^3} \approx \frac{\ell^2}{(G_N M)^3}$$

X ray binary: orbital period derivative

A0620-00:  $10 \pm 5 M_{\odot}$  BH+K-star (Johannsen, Psaltis, McClintock (2009))

$\ell < 0.132\text{mm}$  (assuming  $10 M_{\odot}$  BH )

J1118+480:  $6.8 \pm 0.4 M_{\odot}$  BH+K~M-star (Johannsen (2009))

$\ell < 0.97\text{mm}$  (assuming  $6.8 M_{\odot}$  BH )

In globular cluster RZ2109 located in an elliptical galaxy NGC4472 in the Virgo cluster

Compact object with rapid variability and broad emission line  $\sim 2000\text{km/s}$

➡ BH accreting at around Eddington limit ➡ From the luminosity  $\sim 10 M_{\odot}$  BH

(Zepf, Stern, MacCarone, Kundu, Kamionkowski, Rhode, Salzer, Ciardullo & Gronwall (2008))

Assuming that the association with the globular cluster means old age of BH.  $\sim 10 \pm 3\text{Gyr}$ . ➡  $\ell < 0.003\text{mm}$

(Gnedin, MacCarone, Psaltis, Zepf (2009))

# Summary of Lecture II

- Ultra-spinning black holes will have variety of phases. Their dynamical evolution is also required to be investigated.
- Large black holes can also have peculiar feature in some brane world models, although not so much is understood.
  - Warped extra-dimension model may lead to very fast BH evaporation at the classical level.
  - Again numerical construction of stationary solutions will provide us interesting new branches of black object solutions.