

*Lattice study of the entanglement entropy
in $SU(3)$ Yang-Mills theory*

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— Outline —

- ▶ Introduction - entanglement entropy
- ▶ How to measure the entanglement entropy
- ▶ Lattice QCD simulations
- ▶ Summary and outlook

— Entanglement entropy —

- ▶ Measures how much a given quantum state is **entangled quantum mechanically**.
- ▶ **A non-local quantity** like the Wilson loop as opposed to correlation functions
- ▶ We can probe the quantum properties of the ground state for a quantum system (quantum spin system, quantum Hall liquid, ...).
- ▶ Proportional to **the degrees of freedom**
- ▶ The entanglement entropy can be used **an order parameter**

— Entanglement entropy : definition —

- ▶ Divide a system into two subsystems A and B
- ▶ The density matrix on A is defined by

$$\rho_A = \text{Tr}_B(\rho_{AB}),$$

i.e., by tracing over the states of the subsystem B .

- ▶ ρ_A can be regarded as the density matrix for an observer who can only access to the subsystem A .

entanglement entropy as the von Neumann entropy

$$S_A = -\text{Tr}_A(\rho_A \ln \rho_A)$$

— Entanglement entropy : a simple example —

- ▶ Consider two spin 1/2 particles ($|a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$)

$$|\psi\rangle = a \begin{pmatrix} 1 \\ 0 \end{pmatrix}_A \begin{pmatrix} 1 \\ 0 \end{pmatrix}_B + b \begin{pmatrix} 1 \\ 0 \end{pmatrix}_A \begin{pmatrix} 0 \\ 1 \end{pmatrix}_B + c \begin{pmatrix} 0 \\ 1 \end{pmatrix}_A \begin{pmatrix} 1 \\ 0 \end{pmatrix}_B + d \begin{pmatrix} 0 \\ 1 \end{pmatrix}_A \begin{pmatrix} 0 \\ 1 \end{pmatrix}_B$$

- ▶ The density matrix $\rho_{AB} = |\psi\rangle\langle\psi|$ is given by

$$\rho_{AB} = aa^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_A \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_B + ab^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_A \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_B + \dots$$

- ▶ The reduced density matrix ρ_A is

$$\rho_A = \text{Tr}_B(\rho_{AB}) = \begin{pmatrix} aa^* + bb^* & ac^* + bd^* \\ ca^* + db^* & cc^* + dd^* \end{pmatrix}$$

— Entanglement entropy : a simple example —

- ▶ The eigenvalues of the density matrix ρ_A are

$$\lambda_{\pm} = \frac{1}{2} \left(1 \pm \sqrt{1 - 4|ad - bc|^2} \right)$$

- ▶ The entanglement entropy is

$$S_A = -\text{Tr}_A(\rho_A \ln \rho_A) = -\sum_i \lambda_i \ln \lambda_i$$

- ▶ For a pure state, e.g.,

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_A + |\downarrow\rangle_A \right) \otimes |\uparrow\rangle_B \implies S_A = 0$$

- ▶ For a mixed state, e.g.,

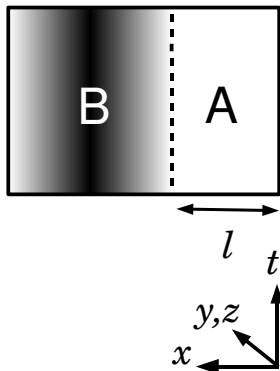
$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B \right) \implies S_A = \ln 2$$

— Entanglement entropy in a quantum field theory —

- ▶ Divide spacetime into two regions A and B
- ▶ Entanglement **between two regions**
= correlations between separated systems
- ▶ ρ_A of the ground state of the theory

$$\rho_A = \text{Tr}_B \rho = \text{Tr}_B |0\rangle\langle 0|$$

- ▶ The entanglement entropy
$$S_A(l) = -\text{Tr}_A(\rho_A \ln \rho_A)$$
- ▶ How to calculate the entanglement entropy?
 1. **replica trick** [Carabrese and Cardy, 2004]
 2. **holography** [Ryu and Takayanagi, 2006]



— Replica trick —

- ▶ The entanglement entropy

$$S_A(l) = -\text{Tr}_A(\rho_A \ln \rho_A)$$

can be written as ($\text{Tr}_A \rho_A = 1$)

$$S_A(l) = -\lim_{n \rightarrow 1} \frac{\partial}{\partial n} \ln \text{Tr}_A \rho_A^n$$

- ▶ The total density matrix ρ is ($Z(T) = \text{Tr} \exp(-\beta H)$)

$$\rho[\phi''(\vec{x}), \phi'(\vec{x})] = Z^{-1}(T) \langle \phi''(\vec{x}) | \exp(-\beta H) | \phi'(\vec{x}) \rangle,$$

or, in the path integral expression,

$$\rho[\phi''(\vec{x}), \phi'(\vec{x})] = Z^{-1}(T) \int_{\phi(\vec{x}, t=0)=\phi'(\vec{x})}^{\phi(\vec{x}, t=\beta)=\phi''(\vec{x})} \mathcal{D}\phi \exp(-S_E)$$

— Replica trick —

- ▶ The total density matrix

$$\begin{aligned}\rho[\phi''(\vec{x}), \phi'(\vec{x})] &= Z^{-1}(T) \langle \phi''(\vec{x}) | \exp(-\beta H) | \phi'(\vec{x}) \rangle \\ &= Z^{-1}(T) \int_{\phi(\vec{x}, t=0)=\phi'(\vec{x})}^{\phi(\vec{x}, t=\beta)=\phi''(\vec{x})} \mathcal{D}\phi \exp(-S_E)\end{aligned}$$

- ▶ $Z(T) = \text{Tr} \exp(-\beta H)$ is found by setting $\phi'(\vec{x}) = \phi''(\vec{x})$ and integrating over $\phi'(\vec{x})$.
- ▶ The reduced density matrix $\rho_A[\phi''(\vec{x}), \phi'(\vec{x})] = \text{Tr}_B \rho$ can be obtained by imposing the boundary conditions

$$\begin{aligned}\phi(\vec{x}, t=0) &= \phi'(\vec{x}) \quad \text{and} \quad \phi(\vec{x}, t=\beta) = \phi''(\vec{x}) \quad \text{if} \quad \vec{x} \in A \\ \phi(\vec{x}, t=0) &= \phi(\vec{x}, t=\beta) \quad \text{if} \quad \vec{x} \in B\end{aligned}$$

— Replica trick —

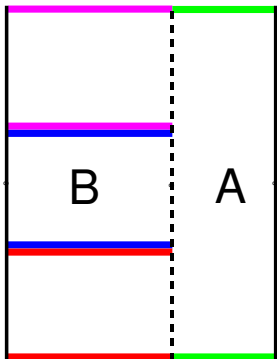
- ▶ n -th power of the reduced density matrix

$$\begin{aligned}\rho_A^n[\phi''(\vec{x}), \phi'(\vec{x})] &= \int_{x \in A} \mathcal{D}\phi_1 \cdots \phi_{n-1} \\ &\rho_A[\phi''(\vec{x}), \phi_1(\vec{x})] \rho_A[\phi_1(\vec{x}), \phi_2(\vec{x})] \cdots \\ &\quad \times \rho_A[\phi_{n-1}(\vec{x}), \phi'(\vec{x})]\end{aligned}$$

- ▶ The trace of ρ_A^n is found to be

$$\begin{aligned}\text{Tr} \rho_A^n &= \int_{x \in A} \mathcal{D}\phi_1 \cdots \phi_n \\ &\rho_A[\phi_1(\vec{x}), \phi_2(\vec{x})] \rho_A[\phi_2(\vec{x}), \phi_3(\vec{x})] \cdots \\ &\quad \times \rho_A[\phi_n(\vec{x}), \phi_1(\vec{x})],\end{aligned}$$

- ▶ $\text{Tr} \rho_A^n$ is obtained by imposing the periodic boundary condition in time with period $n\beta$ if $x \in A$ and with period β if $x \in B$.



— Replica trick —

- ▶ We finally get

$$\text{Tr } \rho_A^n = \frac{Z(l, n, T)}{Z^n(T)}, \quad S_A(l) = - \lim_{n \rightarrow 1} \frac{\partial}{\partial n} \ln \left(\frac{Z(l, n, T)}{Z^n(T)} \right)$$

where $Z(l, n, T)$ is the partition function on the n -sheeted Riemann surface.

- ▶ For $(1 + 1)$ -dimensional CFT,

[Holzhey, Larsen and Wilczek, 1994, Carabrese and Cardy, 2004]

$$S_A(l) = \frac{c}{3} \log \left[\frac{L}{\pi a} \sin \left(\frac{\pi l}{L} \right) \right],$$

where c is the central charge of CFT, a the UV cutoff, and L the length of the total system.

— Entanglement entropy : holographic approach —

- ▶ AdS/CFT correspondence argues that the supergravity on $(d + 2)$ -dimensional anti-de Sitter space AdS_{d+2} is equivalent to a $(d + 1)$ -dimensional conformal field theory living on the boundary of AdS_{d+2} [Maldacena, 1998].
- ▶ It has been proposed that the entanglement entropy S_A in $(d + 1)$ -dimensional CFT can be computed from the 'area law' [Ryu and Takayanagi, 2006]

$$S_A(l) = \frac{\text{area of } \gamma_A}{4G_N^{(d+2)}},$$

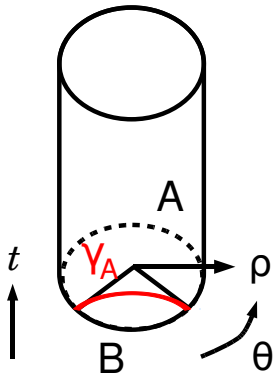
where γ_A is the d -dimensional static minimal surface in AdS_{d+2} with $\partial\gamma_A = \partial A$, and $G_N^{(d+2)}$ is the $(d + 2)$ -dimensional Newton constant.

— Holographic approach : example —

- ▶ The gravitational theories on AdS_3 space of radius R are dual to $(1 + 1)$ -dimensional CFTs with the central charge $c = 3R/2G_N^3$
- ▶ The metric of AdS_3

$$ds^2 = R^2(-\cosh \rho^2 dt^2 + d\rho^2 + \sinh \rho^2 d\theta^2)$$

- ▶ The subsystem A is the region $0 \leq \theta \leq 2\pi l/L$.
- ▶ γ_A is the static geodesic which connects the boundary of A , i.e., the points $\theta = 0$ and $2\pi l/L$ at fixed t traveling inside AdS_3 .



— Holographic approach : example —

- ▶ The geodesic distance L_{γ_A} is given by

$$\cosh\left(\frac{L_{\gamma_A}}{R}\right) = 1 + 2 \sinh^2 \rho_0 \sin^2 \frac{\pi l}{L}$$

- ▶ Assuming $\exp(\rho_0) \gg 1$, the entanglement entropy is

$$S_A(l) \simeq \frac{c}{3} \log \left[\exp(\rho_0) \sin\left(\frac{\pi l}{L}\right) \right], \quad \exp(\rho_0) \sim L/a,$$

which coincides with the result obtained by using the replica trick.

- ▶ In $(3+1)$ -dimensional $\mathcal{N} = 4$ SYM considered to be dual to $\text{AdS}_5 \times S^5$ background in type IIB string theory [Ryu and Takayanagi, 2006]

$$\frac{1}{\partial A} S_A(l) \approx \frac{N_c^2}{a^2} - \frac{N_c^2}{l^2}$$

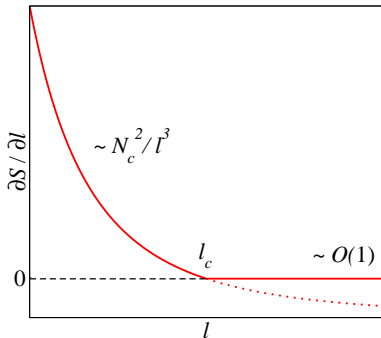
— holographic approach : confining background —

- ▶ Confining large N_c gauge theories dual to supergravity on certain SUSY-breaking 5-dimensional geometries

[Klebanov, Kutasov, and Murugan, 2008]

- ▶ **nonanalytic behavior** has been found for following backgrounds

- ▶ Klebanov-Strassler
- ▶ D4-branes on a circle
- ▶ D3-branes on a circle
($(2+1)$ -dimensional theory)
- ▶ Soft wall model (e^{-z^2} dilaton)



- ▶ Entanglement entropy transition at $l_c \approx m_{\text{glueball}}^{-1}$ from $O(N_c^2)$ solution (**gluons**) at small l to $O(1)$ solution (**glueballs**) at large l

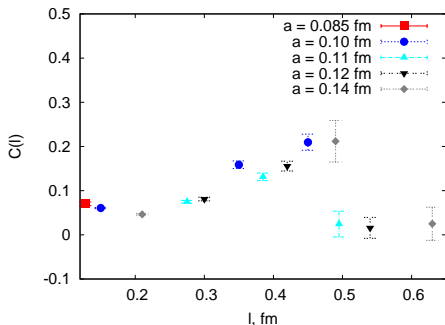
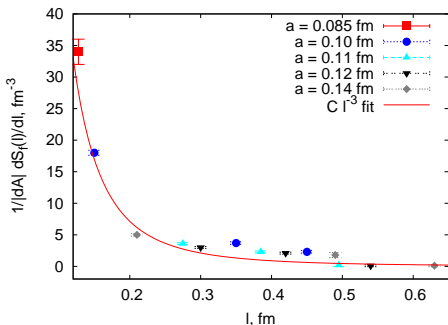
— Entanglement entropy in the lattice gauge theory —

- ▶ Entanglement entropy

$$S_A(l) = - \lim_{n \rightarrow 1} \frac{\partial}{\partial n} \frac{Z(l, n, T)}{Z^n(T)},$$

- ▶ (Exactly solvable) (1 + 1)-dimensional $SU(N_c)$ lattice gauge theory analytic behavior (independent of l)
- ▶ (3 + 1)-dimensional $SU(2)$ lattice gauge theory treated within Migdal-Kadanoff approximation
 \Rightarrow nonanalytic change at $l_c T_c \in (1.56, 1.66)$ [Velytsky, 2008]

— Entanglement entropy in SU(2) lattice gauge theory —



[Buividovich and Polikarpov, 2008]

- ▶ entropic C-function

$$C(l) = \frac{l^d}{\partial A} \frac{\partial S_A(l)}{\partial l}$$

measures the effective degrees of freedom at the energy scale $\Lambda \sim 1/l$

— Lattice QCD simulations —

- ▶ SU(3) quenched lattice simulations
- ▶ Pseudo heat-bath MC update: Wilson plaquette action

$$S_W = \beta \sum_p \left(1 - \frac{1}{2N_c} \text{Tr}(U_p + U_p^\dagger) \right)$$

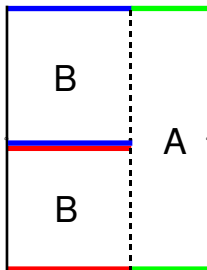
- ▶ Measure the derivative of $S_A(l)$ with respect to l

$$\begin{aligned} \frac{\partial S_A(l)}{\partial l} &= \frac{\partial}{\partial l} \left[- \lim_{n \rightarrow 1} \frac{\partial}{\partial n} \ln \left(\frac{Z(l, n, T)}{Z^n(T)} \right) \right] \\ &= \lim_{n \rightarrow 1} \frac{\partial}{\partial l} \frac{\partial}{\partial n} F[l, n, T] \end{aligned}$$

— Lattice QCD simulations —

- ▶ Estimate the derivative by

$$\begin{aligned}\frac{\partial S_A(l)}{\partial l} &= \lim_{n \rightarrow 1} \frac{\partial}{\partial l} \frac{\partial}{\partial n} F[A, n, T] \\ &\rightarrow \frac{\partial}{\partial l} \lim_{n \rightarrow 1} (F[l, n+1, T] - F[l, n, T]) \\ &\rightarrow \frac{F[l+a, 2, T] - F[l, 2, T]}{a}\end{aligned}$$



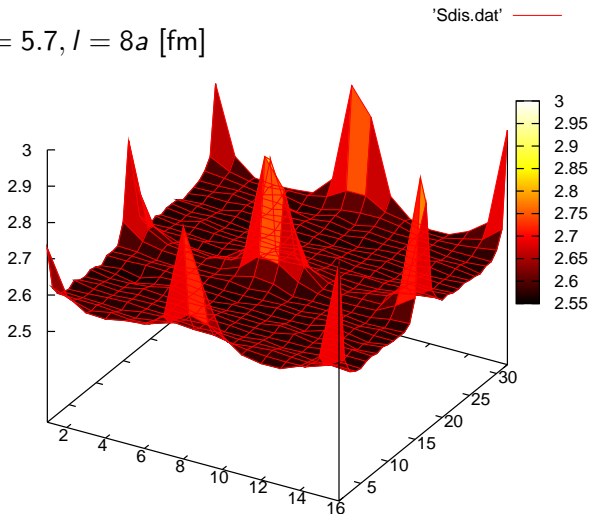
- ▶ Differences of free energies [Endrodi et al., PoS LAT2007]

$$Z(\alpha) = \int \mathcal{D}\phi \exp(-(1-\alpha)S_1[\phi] - \alpha S_2[\phi])$$

$$F_2 - F_1 = - \int_0^1 d\alpha \frac{\partial}{\partial \alpha} \ln Z(\alpha) = \int_0^1 d\alpha \langle S_2[\phi] - S_1[\phi] \rangle_\alpha$$

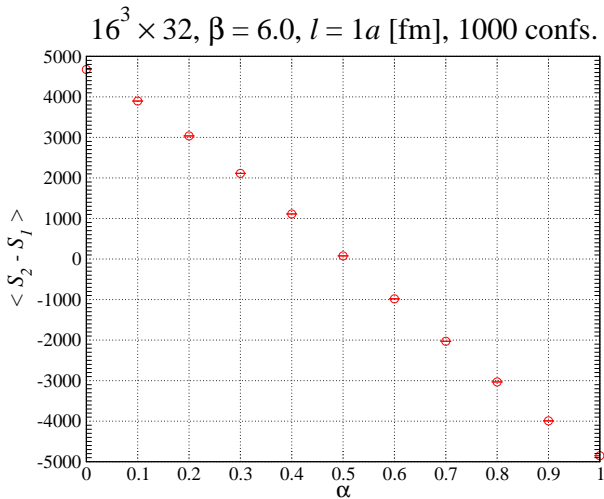
— Results: average action density —

- $16^3 \times 32, \beta = 5.7, l = 8a$ [fm]



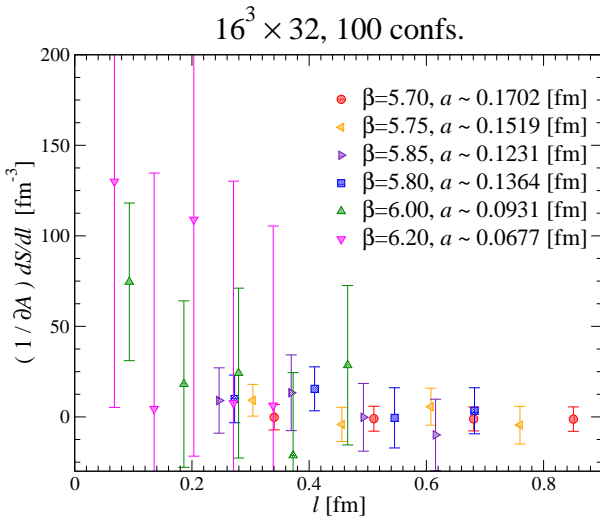
- ▶ The action density is larger in the vicinity of the branching points.

— Results: the action difference —



- ▶ Contribution from $\alpha > 0.5$ and $\alpha < 0.5$ almost cancel exactly.

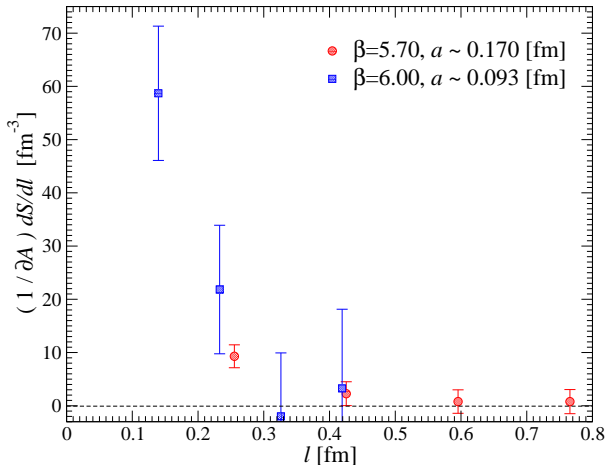
— Results: derivative of $S_A(l)$ wrt l —



► Large statistical errors, increases with the lattice coupling

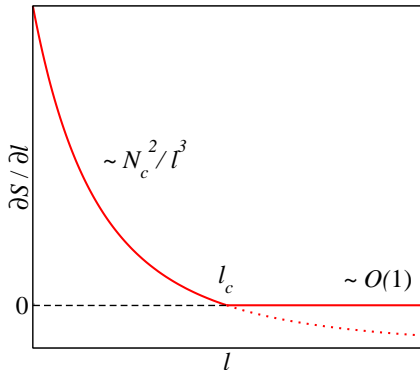
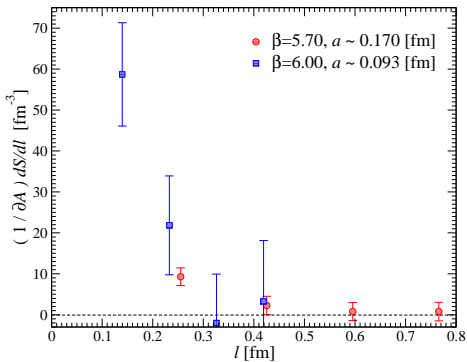
— Results: derivative of $S_A(l)$ wrt l —

$16^3 \times 32, 1000 \sim 1300$ confs.



— Results: comparison with AdS/CFT result —

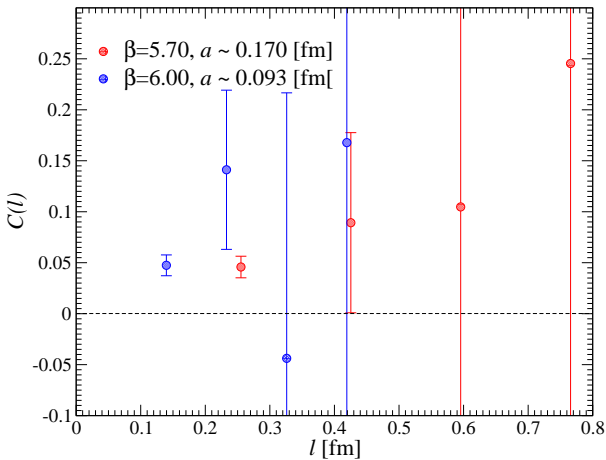
$16^3 \times 32$, 1000 ~ 1300 confs.



► discontinuity at $l_c \approx 0.3 \sim 0.4$ [fm]??

— Results: entropic C -function —

$16^3 \times 32, 1000$ confs.



► Very noisy. Vanishes above $l > 0.3$ [fm]? Need more statistics!

— Summary and outlook —

- ▶ We discussed the entanglement entropy in SU(3) pure YM theory.
 - It measures the **effective d.o.f. at the energy scale $\Lambda \sim 1/l$** .
 - It may be expected that the entanglement entropy serves as an order parameter for **asymptotic freedom/confinement phase transition**.
 - We observe a **sign of the discontinuity** at about $l \approx 0.3 \sim 0.4$ [fm].
 - The entropic C -function is very noisy, and we need more statistics.

— Summary and outlook —

- ▶ We have to examine
 - **finite volume effects** (calculations on large lattice volume)
 - **discretization errors** (with an improved gauge action)
- ▶ It is very interesting to see if we observe the discontinuity in the entanglement entropy
 - in the **deconfinement phase**
 - in **full QCD**

— Analyticity of $\text{Tr } \rho_A$ —

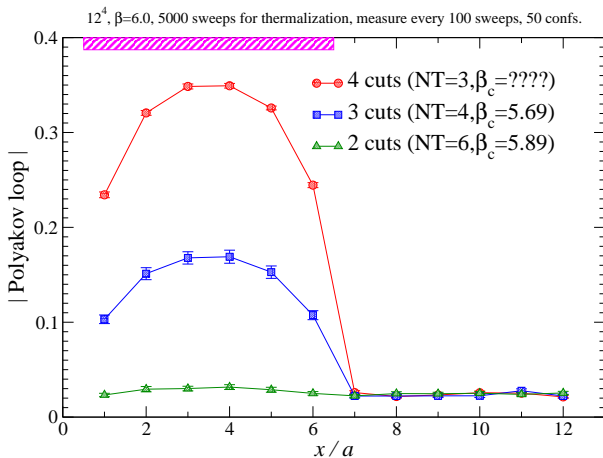
See [Calabrese and Cardy, quant-ph/0505193]

$$\text{Tr } \rho_A^n = \frac{Z(l, n, T)}{Z^n(T)},$$

$$\text{Tr } \rho_A^n = \sum_i \lambda_i^n, \quad 0 \leq \lambda_i < 1$$

- ▶ $\text{Tr } \rho_A^n$ is absolutely convergent and analytic for all $\Re n > 1$.
- ▶ The derivative with respect to n exists and analytic in the region.
- ▶ If $\rho_A = -\sum_i \lambda_i \ln \lambda_i$ is finite, then the limit as $n \rightarrow 1^+$ of the first derivative converges to ρ_A .
- ▶ $Z(l, n, T)/Z^n(T)$ has a unique analytic continuation to $\Re n > 1$.

— Results: Polyakov loop with the cuts —



— Results: Polyakov loop winds around the region B —

