Relativistic Radiation Hydrodynamics for Accretion

Myeong-Gu Park
Kyungpook National University, Daegu, KOREA

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Radiation and Matter

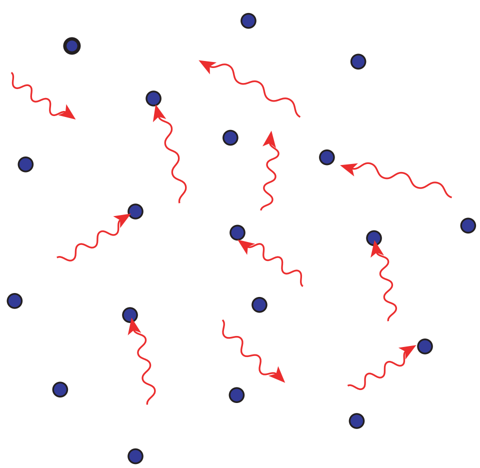
- Radiation always interacts with matter, via exchange of energy and momentum
  - matter: *local*

$$\rho(\boldsymbol{x}, t), \quad T(\boldsymbol{x}, t), \quad \boldsymbol{v}(\boldsymbol{x}, t)$$

  - radiation: *global*

$$\text{specific intensity} \quad I_\nu = I(\boldsymbol{x}, t; \nu, \boldsymbol{\Omega})$$

- Matter and radiation field should be solved together!



Non-relativistic Radiation Hydrodynamics

- Spherically Symmetric Case

  – Continuity equation

$$\frac{\partial n}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 n v^r) = 0$$

  – Euler equation

$$\rho\left(\frac{\partial v^r}{\partial t} + v^r \frac{\partial v^r}{\partial r}\right) + \rho\frac{GM}{r^2} + \frac{\partial P_g}{\partial r} = \underline{\bar{\chi}F^r}$$

  – Gas energy equation

$$-n\frac{\partial}{\partial t}\left(\frac{\omega_g}{n}\right) - n v^r \frac{\partial}{\partial r}\left(\frac{\omega_g}{n}\right) + \frac{\partial P_g}{\partial t} + v^r \frac{\partial P_g}{\partial r} = \underline{\Gamma - \Lambda}$$

– Radiation energy equation

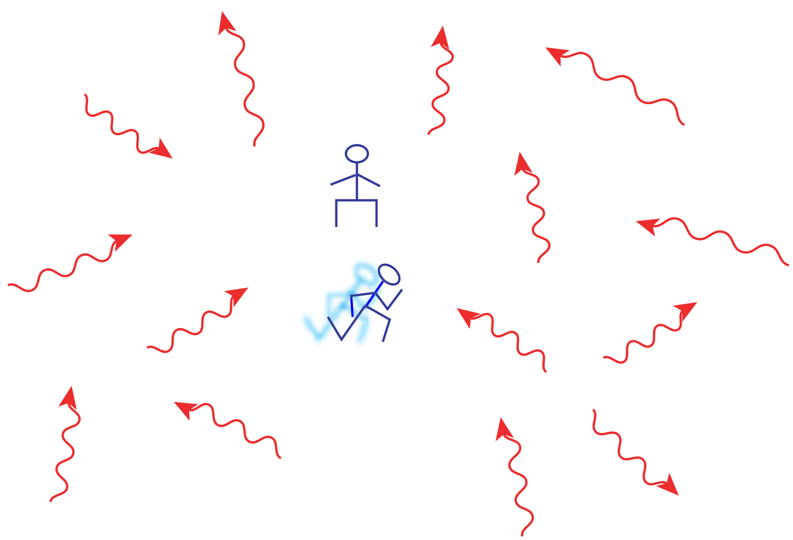
$$\frac{\partial E}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F^r\right) = \underline{\Lambda - \Gamma}$$

– Radiation momentum equation

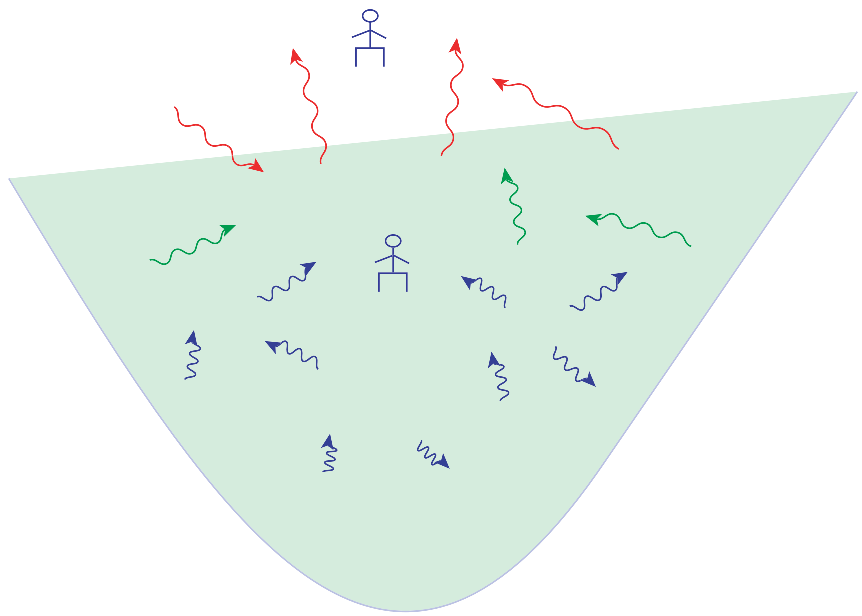
$$\frac{\partial F^r}{\partial t} + \frac{\partial P}{\partial r} - \frac{E - 3P}{r} = -\underline{\bar{\chi}_{co} F^r}$$

Relativistic Effects

- Observer dependent measurement
  - red/blue shift
  - relativistic beaming
  - time dilation
  - bulk Comptonization



- Spacetime curvature
  - gravitational redshift $\longrightarrow$ frequency mixing
  - gravitational time dilation
  - loss cone
  - gravitational light bending



Previous Works

- Thomas (1930):
  special relativistic theory of radiative transfer in diffusion limit
- Lindquist (1966):
  theory of radiative transfer in curved spacetime
  covariant moment equation under spherical symmetry
- Anderson & Spiegel (1972):
  generalized moment equation including the second moments
- Thorne (1981):
  Projected Symmetric Trace-Free moment formalism
  comoving proper frame
- Park (1993):
  mixed-frame moment equations under spherical symmetry

Steps to Construct Relativistic RHD

1. Define relevant physical variables in covariant forms
2. Construct covariant conservation equations
3. Choose specific coordinates or spacetime
    - Define variables in a suitable tetrad
    - Establish the transformation between tetrad and coordinates
    - Transform the variables to covariant ones
4. Derive the equations in desirable forms

Energy-momentum Tensors

- Energy-momentum tensor of gas

$$T^{\alpha\beta} \equiv \omega_g U^\alpha U^\beta + P_g g^{\alpha\beta}$$

gas enthalpy: $\omega_g \equiv \varepsilon_g + P_g = nmc^2 + \ldots$

- Radiation stress tensor

$$R^{\alpha\beta} = \int\int I(\boldsymbol{n}, \nu) n^\alpha n^\beta d\nu d\Omega$$

$n^\alpha \equiv p^\alpha / h\nu$

I_ν/ν^3, $\nu d\nu d\Omega$: frame independent scalars

- Radiation four-force density [Mihalas & Mihalas 1984]

$$G^\alpha \equiv \frac{1}{c} \int d\nu \int d\Omega [\chi I(\boldsymbol{n}, \nu) - \eta] n^\alpha$$

in comoving frame: $G^{\hat{t}}_{co} = \Gamma_{co} - \Lambda_{co}, \quad G^{\hat{i}}_{co} = \bar{\chi}_{co} F^i_{co}$

Conservation Equations

- Particle number conservation

$$(nU^{\alpha})_{;\alpha} = 0$$

- Energy-momentum conservation

$$\left(T^{\alpha\beta} + R^{\alpha\beta}\right)_{;\beta} = f^{\alpha}.$$

or if the interactions between gas and radiation are known

$$T^{\alpha\beta}{}_{;\beta} - f^{\alpha} = G^{\alpha} = -R^{\alpha\beta}{}_{;\beta}$$

Radiation Hydrodynamic Equations

- Euler Equation

$$P_\lambda{}^\alpha (T^{\lambda\mu}{}_{;\mu} - f^\lambda) = P_\lambda{}^\alpha G^\lambda \quad \text{where} \quad P_\alpha{}^\beta = \delta_\alpha{}^\beta + U_\alpha U^\beta$$

- Energy Equation

$$U_\alpha (T^{\alpha\beta}{}_{;\beta} - f^\alpha) = U_\alpha G^\alpha$$

- Radiation Moment Equation

$$R^{\alpha\beta}{}_{;\beta} = -G^\alpha$$

Tetrads and Velocities

- Basis vectors and orthonormal tetrads

$$\frac{\partial}{\partial x^{\alpha}}; \quad \frac{\partial}{\partial x^{\hat{\alpha}}} \equiv \frac{1}{\sqrt{g_{\alpha\alpha}}}\frac{\partial}{\partial x^{\alpha}}; \quad \frac{\partial}{\partial x^{\hat{\alpha}}_{co}} = \underline{\Lambda^{\hat{\beta}}_{\hat{\alpha}}(\boldsymbol{v})}\frac{\partial}{\partial x^{\hat{\beta}}}$$

- Four velocity

$$U^{\alpha} \equiv \frac{dx^{\alpha}}{d\tau}; \quad U_{\alpha}U^{\alpha} = -1; \quad \tilde{u}^{i} \equiv \sqrt{g_{ii}}\,U^{i}$$

- Proper velocity
  - measured by the fiducial observer at rest

$$v^{i} = \frac{U^{\hat{i}}}{U^{\hat{t}}} \quad (i = 1, 2, 3)$$

- Lorentz factor

$$\gamma \equiv (1 - v^{i}v_{i})^{-1/2}$$

- Lorentz transformation

$$\begin{aligned}
\Lambda^{\hat{t}}{}_{\hat{t}} &= \gamma, \\
\Lambda^{\hat{i}}{}_{\hat{t}} &= \gamma v^{i}, \quad \Lambda^{\hat{t}}{}_{\hat{j}} = \gamma v_{j}, \\
\Lambda^{\hat{i}}{}_{\hat{j}} &= \delta^{i}{}_{j} + v^{i} v_{j} \frac{\gamma - 1}{v^{2}}
\end{aligned}$$

- From fixed tetrad to comoving tetrad

$$\frac{\partial}{\partial x_{co}^{\hat{\alpha}}} = \Lambda^{\hat{\beta}}{}_{\hat{\alpha}}(\boldsymbol{v}) \frac{\partial}{\partial x^{\hat{\beta}}}$$

Radiation Moments

- Radiation energy density

$$E = \iint I_\nu d\nu d\Omega, \quad E_{co} = \iint I_{\nu co} d\nu_{co} d\Omega_{co}$$

- Radiation flux

$$F^i = \iint I_\nu n^i d\nu d\Omega, \quad F^i_{co} = \iint I_{\nu co} n^i_{co} d\nu_{co} d\Omega_{co}$$

- Radiation pressure

$$P^{ij} = \iint I_\nu n^i n^j d\nu d\Omega, \quad P^{ij}_{co} = \iint I_{\nu co} n^i_{co} n^j_{co} d\nu_{co} d\Omega_{co}$$

- Transformation between fixed and comoving moments

$$R_{co}^{\hat{a}\hat{b}} = \Lambda^{\hat{\alpha}}{}_{\hat{\lambda}}(-\boldsymbol{v})\Lambda^{\hat{\beta}}{}_{\hat{\mu}}(-\boldsymbol{v})R^{\hat{\lambda}\hat{\mu}}$$

$$\begin{aligned}
E_{co} &= \gamma^2\left[\underline{E} - \underline{2v_iF^i} + v_iv_jP^{ij}\right] \\
F_{co}^i &= \left[\underline{\delta_j^i} + (\frac{\gamma-1}{v^2} + \gamma^2)v^iv_j\right]F^j - \gamma^2\underline{v^iE} \\
&\quad -\gamma\underline{v_j}\left[\underline{\delta_k^i} + \frac{\gamma-1}{v^2}v^iv_k\right]\underline{P^{jk}} \\
P_{co}^{ij} &= \gamma^2v^iv^jE - \gamma\left[\underline{v^i\delta_k^j} + \underline{v^j\delta_k^i} + 2\frac{\gamma-1}{v^2}v^iv^jv_k\right]\underline{F^k} \\
&\quad +(\underline{\delta_k^i} + \frac{\gamma-1}{v^2}v^iv_k)(\underline{\delta_l^j} + \frac{\gamma-1}{v^2}v^jv_l)P^{kl}
\end{aligned}$$

Radiation Stress Tensor

- Radiation stress tensor: fixed and comoving tetrad form

$$R^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} E & F^i \\ F^j & P^{ij} \end{pmatrix}, \quad R_{co}^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} E_{co} & F_{co}^i \\ F_{co}^j & P_{co}^{ij} \end{pmatrix}$$

- Radiation stress tensor: covariant form

$$R^{\alpha\beta} = \frac{\partial x^\alpha}{\partial x^{\hat{\lambda}}} \frac{\partial x^\beta}{\partial x^{\hat{\mu}}} R^{\hat{\lambda}\hat{\mu}}$$

Schwarzschild Spacetime

- Metric

$$
\begin{aligned}
d\tau^2 &= -g_{\alpha\beta}dx^\alpha dx^\beta \\
&= \Gamma^2 dt^2 - \frac{dr^2}{\Gamma^2} - r^2(d\theta^2 + \sin^2\theta d\phi^2)
\end{aligned}
$$

$$\Gamma^2 \equiv 1 - 2m/r, \quad m \equiv GM/c^2, \quad c \equiv 1$$

- Four-velocity

$$U^\alpha \equiv \frac{dx^\alpha}{d\tau}; \quad U^\alpha U_\alpha = -1$$

- Energy parameter

$$
\begin{aligned}
y &\equiv -U_t \\
&= \left[\Gamma^2 + (U^r)^2 + \Gamma^2\{(rU^\theta)^2 + (r\sin\theta U^\phi)^2\}\right]^{1/2}
\end{aligned}
$$

- Fixed tetrad

$$\frac{\partial}{\partial \hat{t}} = \frac{1}{\Gamma}\frac{\partial}{\partial t}; \quad \frac{\partial}{\partial \hat{r}} = \Gamma\frac{\partial}{\partial r}$$
$$\frac{\partial}{\partial \hat{\theta}} = \frac{1}{r}\frac{\partial}{\partial \theta}; \quad \frac{\partial}{\partial \hat{\phi}} = \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

- Proper velocity $\boldsymbol{v}$

$$v^r = \frac{1}{y}U^r; \quad v^\theta = \frac{\Gamma}{y}rU^\theta; \quad v^\phi = \frac{\Gamma}{y}r\sin\theta U^\phi$$

- Lorentz factor

$$\gamma \equiv (1 - v^2)^{-1/2} = \frac{y}{\Gamma}$$

- Comoving tetrad vs coordinate base

$$
\begin{aligned}
\frac{\partial}{\partial \hat{t}_{co}} &= \frac{\gamma}{\Gamma}\frac{\partial}{\partial t} + \gamma\Gamma v_r \frac{\partial}{\partial r} + \gamma v_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \gamma v_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \\
\frac{\partial}{\partial \hat{r}_{co}} &= \frac{\gamma}{\Gamma} v_r \frac{\partial}{\partial t} + \Gamma\left[1 + (\gamma - 1)\frac{v_r^2}{v^2}\right]\frac{\partial}{\partial r} \\
&+ (\gamma - 1)\frac{v_r v_\theta}{v^2}\frac{1}{r}\frac{\partial}{\partial\theta} + (\gamma - 1)\frac{v_r v_\phi}{v^2}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \\
\frac{\partial}{\partial \hat{\theta}_{co}} &= \frac{\gamma}{\Gamma} v_\theta \frac{\partial}{\partial t} + \Gamma(\gamma - 1)\frac{v_r v_\theta}{v^2}\frac{\partial}{\partial r} \\
&+ \left[1 + (\gamma - 1)\frac{v_\theta^2}{v^2}\right]\frac{1}{r}\frac{\partial}{\partial\theta} + (\gamma - 1)\frac{v_\theta v_\phi}{v^2}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \\
\frac{\partial}{\partial \hat{\phi}_{co}} &= \frac{\gamma}{\Gamma} v_\phi \frac{\partial}{\partial t} + \Gamma(\gamma - 1)\frac{v_r v_\phi}{v^2}\frac{\partial}{\partial r} \\
&+ (\gamma - 1)\frac{v_\theta v_\phi}{v^2}\frac{1}{r}\frac{\partial}{\partial\theta} + \left[1 + (\gamma - 1)\frac{v_\phi^2}{v^2}\right]\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}
\end{aligned}
$$

Radiation Stress Tensor

- Fixed tetrad components

$$R^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} E & F^r & F^\theta & F^\phi \\ F^r & P^{rr} & P^{r\theta} & P^{r\phi} \\ F^\theta & P^{r\theta} & P^{\theta\theta} & P^{\theta\phi} \\ F^\phi & P^{r\phi} & P^{\theta\phi} & P^{\phi\phi} \end{pmatrix}$$

  - Spherically symmetric case:

$$R^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} E & F^r & 0 & 0 \\ F^r & P^{rr} & 0 & 0 \\ 0 & 0 & 2^{-1}(E - P^{rr}) & 0 \\ 0 & 0 & 0 & 2^{-1}(E - P^{rr}) \end{pmatrix}$$

- Comoving tetrad components

$$R_{co}^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} E_{co} & F_{co}^{r} & F_{co}^{\theta} & F_{co}^{\phi} \\ F_{co}^{r} & P_{co}^{rr} & P_{co}^{r\theta} & P_{co}^{r\phi} \\ F_{co}^{\theta} & P_{co}^{r\theta} & P_{co}^{r\theta} & P_{co}^{\theta\phi} \\ F_{co}^{\phi} & P_{co}^{r\phi} & P_{co}^{\theta\phi} & P_{co}^{\phi\phi} \end{pmatrix}$$

- Covariant form

$$R^{\alpha\beta} = \begin{pmatrix} \Gamma^2 E & F^r & \Gamma^{-1}\frac{F^\theta}{r} & \Gamma^{-1}\frac{F^\phi}{r\sin\theta} \\ F^r & \Gamma^2 P^{rr} & \Gamma\frac{P^{r\theta}}{r} & \Gamma\frac{P^{r\phi}}{r\sin\theta} \\ \Gamma^{-1}\frac{F^\theta}{r} & \Gamma\frac{P^{r\theta}}{r} & \frac{P^{\theta\theta}}{r^2} & \frac{P^{\theta\phi}}{r^2\sin\theta} \\ \Gamma^{-1}\frac{F^\phi}{r\sin\theta} & \Gamma\frac{P^{r\phi}}{r\sin\theta} & \frac{P^{\theta\phi}}{r^2\sin\theta} & \frac{P^{\phi\phi}}{r^2\sin^2\theta} \end{pmatrix}$$

Radiation Four-Force Density

- Coordinate vs tetrad components

$$G^{\alpha} = \frac{\partial x^{\alpha}}{\partial x_{co}^{\hat{\beta}}} G_{co}^{\hat{\beta}}$$

$$\begin{aligned}
G^{t} &= \frac{1}{\Gamma}\left[\underline{\gamma G_{co}^{\hat{t}}} + \gamma v_i G_{co}^{\hat{i}}\right] \\
G^{r} &= \Gamma\left[\underline{G_{co}^{\hat{r}}} + \gamma v_r G_{co}^{\hat{t}} + \frac{\gamma - 1}{v^2} v_r v_i G_{co}^{\hat{i}}\right] \\
G^{\theta} &= \frac{1}{r}\left[\underline{G_{co}^{\hat{\theta}}} + \gamma v_\theta G_{co}^{\hat{t}} + \frac{\gamma - 1}{v^2} v_\theta v_i G_{co}^{\hat{i}}\right] \\
G^{\phi} &= \frac{1}{r \sin\theta}\left[\underline{G_{co}^{\hat{\phi}}} + \gamma v_\phi G_{co}^{\hat{t}} + \frac{\gamma - 1}{v^2} v_\phi v_i G_{co}^{\hat{i}}\right]
\end{aligned}$$

Hydrodynamic Equations

- Continuity equation: particle number conservation

$$\frac{1}{\Gamma^2}\frac{\partial}{\partial t}(yn) + \frac{1}{r^2}\frac{\partial}{\partial r}(\underline{r^2 n U^r})$$
$$+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta n U^\theta) + \frac{\partial}{\partial\phi}(nU^\phi) = 0$$

  – Spherically symmetric case: $\underline{4\pi r^2 n U^r} = -\dot{N}$

- Euler equation
  - r

$$\begin{aligned}
& \omega_g U^t \frac{\partial U^r}{\partial t} + \underline{\omega_g U^i \frac{\partial U^r}{\partial x^i}} + \omega_g \frac{m}{r^2} \left[\Gamma^2 (U^t)^2 - \Gamma^{-2} (U^r)^2 \right] \\
- \quad & \omega_g \frac{\Gamma^2}{r} \left[(rU^\theta)^2 + (r \sin\theta U^\phi)^2 \right] + U^r U^t \frac{\partial P_g}{\partial t} + \Gamma^2 \frac{\partial P_g}{\partial r} + U^r U^i \frac{\partial P_g}{\partial x^i} \\
= \quad & -yU^r G^t + [1 + \Gamma^{-2}(U^r)^2] G^r + r^2 U^r U^\theta G^\theta + r^2 \sin^2\theta U^r U^\phi G^\phi \\
& + f^r + U^r U_\alpha f^\alpha
\end{aligned}$$

  - Spherically symmetric case without external force:

$$\begin{aligned}
& \frac{y}{\Gamma} \frac{\partial U^r}{\partial t} + \frac{1}{2} \frac{\partial (U^r)^2}{\partial r} + \frac{m}{r^2} + \frac{y}{\Gamma} \frac{U^r}{\omega_g} \frac{\partial P_g}{\partial t} + \frac{y^2}{\omega_g} \frac{\partial P_g}{\partial r} \\
& = \frac{y}{\omega_g} G^{\hat{r}}_{co} = \frac{y}{\omega_g} \underline{\bar{\chi}_{co} F^r_{co}},
\end{aligned}$$

- θ

$$
\begin{aligned}
& \omega_g U^t \frac{\partial U^\theta}{\partial t} + \omega_g U^i \frac{\partial U^\theta}{\partial x^i} + 2\omega_g \frac{1}{r} U^r U^\theta - \omega_g \sin\theta\cos\theta (U^\phi)^2 \\
+ \quad & U^\theta U^t \frac{\partial P_g}{\partial t} + \frac{1}{r^2}\frac{\partial P_g}{\partial \theta} + U^\theta U^i \frac{\partial P_g}{\partial x^i} \\
= \quad & -y U^\theta G^t + \Gamma^{-2} U^r U^\theta G^r + [1 + r^2 (U^\theta)^2] G^\theta + r^2 \sin^2\theta U^\theta U^\phi G^\phi \\
& + f^\theta + U^\theta U_\alpha f^\alpha
\end{aligned}
$$

- ϕ

$$
\begin{aligned}
& \omega_g U^t \frac{\partial U^\phi}{\partial t} + \omega_g U^i \frac{\partial U^\phi}{\partial x^i} + 2\omega_g \frac{1}{r} U^r U^\phi + 2\omega_g \cot\theta U^\theta U^\phi \\
+ \quad & U^\phi U^t \frac{\partial P_g}{\partial t} + \frac{1}{r^2 \sin^2\theta}\frac{\partial P_g}{\partial \phi} + U^\phi U^i \frac{\partial P_g}{\partial x^i} \\
= \quad & -y U^\phi G^t + \Gamma^{-2} U^r U^\phi G^r + r^2 U^\theta U^\phi G^\theta + [1 + r^2 \sin^2\theta (U^\phi)^2] G^\phi \\
& + f^\phi + U^\phi U_\alpha f^\alpha
\end{aligned}
$$

- Energy equation

$$
\begin{aligned}
& - \quad nU^t \frac{\partial}{\partial t}\left(\frac{\omega_g}{n}\right) - nU^i \frac{\partial}{\partial x^i}\left(\frac{\omega_g}{n}\right) + U^t \frac{\partial P_g}{\partial t} + U^i \frac{\partial P_g}{\partial x^i} - U_\alpha f^\alpha \\
& = \quad -yG^t + \Gamma^{-2} U^r G^r + r^2 U^\theta G^\theta + r^2 \sin^2\theta U^\phi G^\phi \\
& = \quad -G^{\hat{t}}_{co} = \Lambda_{co} - \Gamma_{co}.
\end{aligned}
$$

Radiation Moment Equations

$$R^{\alpha\beta}{}_{;\beta} = -G^{\alpha}$$

- Radiation energy equation

$$\frac{1}{\Gamma^2}\frac{\partial E}{\partial t} + \frac{1}{\Gamma^2}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\Gamma^2 F^r) + \frac{1}{\Gamma r \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta F^\theta) + \frac{1}{\Gamma r \sin\theta}\frac{\partial}{\partial \phi}(F^\phi)$$
$$= -G^t = \frac{y}{\Gamma^2}\left(\Lambda_{co} - \Gamma_{co} - \underline{\bar{\chi}_{co} v_i F^i_{co}}\right)$$

  - Spherically symmetric case: gravitational redshift

$$4\pi r^2 \underline{\left(1 - \frac{2m}{r}\right)} F^r = L_\infty$$

- Radiation momentum equation
  - $\alpha = r$

$$\frac{\partial F^r}{\partial t} + \Gamma^2 \frac{\partial P^{rr}}{\partial r} + \frac{\Gamma}{r \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta P^{r\theta} \right) + \frac{\Gamma}{r \sin\theta} \frac{\partial P^{r\phi}}{\partial \phi}$$

$$+ \frac{m}{r^2} \left(E + P^{rr} \right) + \frac{\Gamma^2}{r} \left(2P^{rr} - P^{\theta\theta} - P^{\phi\phi} \right)$$

$$= -G^r$$

$$= -\Gamma \bar{\chi}_{co} F^r_{co} - \Gamma \gamma \underline{v_r (\Gamma_{co} - \Lambda_{co})} - \Gamma \frac{\gamma - 1}{v^2} v_r v_i \bar{\chi}_{co} F^i_{co}$$

  - Static case

$$F^i = -\frac{1}{\bar{\chi}_{co}} P^{ij}{}_{;j}$$

- $\alpha = \theta$

$$\frac{1}{\Gamma}\frac{\partial F^{\theta}}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(\Gamma r P^{r\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta P^{\theta\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial P^{\theta\phi}}{\partial\phi}$$
$$+ \quad \frac{2\Gamma}{r}P^{r\theta} - \frac{1}{r\tan\theta}P^{\phi\phi}$$
$$= \quad -rG^{\theta} = -\bar{\chi}_{co}F^{\theta}_{co} - \gamma \underline{v_{\theta}(\Gamma_{co} - \Lambda_{co})} - \frac{\gamma - 1}{v^2}v_{\theta}v_i\bar{\chi}_{co}F^{i}_{co}$$

- $\alpha = \phi$

$$\frac{1}{\Gamma}\frac{\partial F^{\phi}}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(\Gamma r P^{r\phi}\right) + \frac{1}{r}\frac{\partial P^{\theta\phi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial P^{\phi\phi}}{\partial\phi}$$
$$+ \quad \frac{2\Gamma}{r}P^{r\theta} + \frac{2}{r\tan\theta}P^{\theta\phi}$$
$$= \quad -r\sin\theta G^{\phi} = -\bar{\chi}_{co}F^{\phi}_{co} - \gamma \underline{v_{\phi}(\Gamma_{co} - \Lambda_{co})} - \frac{\gamma - 1}{v^2}v_{\phi}v_i\bar{\chi}_{co}F^{i}_{co}$$

Incompleteness

- Number of Physical Quantities: 16
  n, T, U^α, E, F^i, P^{ij}
- Number of Equations: 10
  $U^\alpha U_\alpha = -1$: 1
  Continuity equation: 1
  Euler equation: 3
  Energy equation: 1
  Radiation moment equation: 4
- Radiation moment equations are not closed!
  Should solve fully angle-dependent radiative transfer equation: -.-;;;
  Terminate higher moments
  Assume closure relation, such as the Eddington factor

Relativistic Radiative Transfer

- Full angle-dependent radiative transfer calculation
  - $I(\boldsymbol{x}, t; \nu, \boldsymbol{\Omega})$
  - 7 dimensional problem: $\underline{\boldsymbol{x}, t, \nu, \boldsymbol{\Omega}}$
  - ν changes
  - photon trajectory is not straight
- Methods
  - Direct integration by finite difference method
  - Characteristic method

Eddington Factor

- a la Minerbo (1978): choose the Eddington factor that maximizes the entropy

$$f^{ij} \equiv \frac{P^{ij}_{co}}{E_{co}}$$

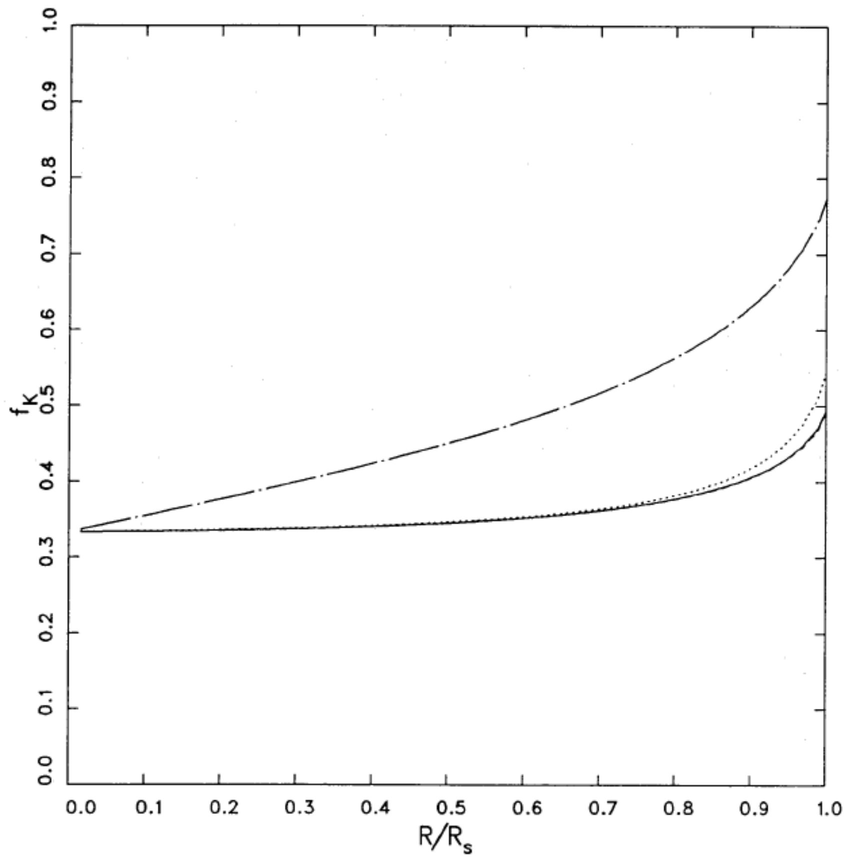
as a function of E_{co} and F^i_{co},

$$f^{ij} = \frac{1}{2}(1 - R_2)\delta^{ij} + \frac{1}{2}(3R_2 - 1)\frac{F^i_{co}F^j_{co}}{F^2_{co}}$$

$$R_2 = 1 - \frac{2}{\lambda}\left(\coth\lambda - \frac{1}{\lambda}\right)$$

$$R_1 \equiv \frac{F_{co}}{E_{co}} = \coth\lambda - \frac{1}{\lambda}$$

- How good is the Eddington factor?



3D Cylindrical Coordinates

- Metric

$$d\tau^2 = dt^2 - dR^2 - R^2 d\theta^2 - dz^2$$

- Tetrad

$$\frac{\partial}{\partial \hat{t}} = \frac{\partial}{\partial t}, \quad \frac{\partial}{\partial \hat{R}} = \frac{\partial}{\partial R}, \quad \frac{\partial}{\partial \hat{\theta}} = \frac{1}{R}\frac{\partial}{\partial \theta}, \quad \frac{\partial}{\partial \hat{z}} = \frac{\partial}{\partial z}$$

- Continuity equation

$$\frac{\partial}{\partial t}(\gamma n) + \frac{1}{R}\frac{\partial}{\partial R}(RnU^R) + \frac{\partial}{\partial \theta}(nU^\theta) + \frac{\partial}{\partial z}(nU^z) = 0,$$

- Euler equation:

$$\begin{aligned}
R: \quad & \gamma\omega_g \frac{\partial U^R}{\partial t} + \omega_g U^i \frac{\partial U^R}{\partial x^i} - \omega_g R (U^\theta)^2 + \frac{\partial P_g}{\partial R} + \gamma U^R \frac{\partial P_g}{\partial t} + U^R U^i \frac{\partial P_g}{\partial x^i} \\
= \quad & -\gamma U^R G^t + [1 + (U^R)^2] G^R + R^2 U^R U^\theta G^\theta + U^R U^z G^z + f^R + U^R U_\beta f^\beta \\
\theta: \quad & \gamma\omega_g \frac{\partial U^\theta}{\partial t} + \omega_g U^i \frac{\partial U^\theta}{\partial x^i} + 2\omega_g \frac{U^R U^\theta}{R} + \frac{1}{R^2}\frac{\partial P_g}{\partial \theta} + \gamma U^\theta \frac{\partial P_g}{\partial t} + U^\theta U^i \frac{\partial P_g}{\partial x^i}
\end{aligned}$$

$$
\begin{aligned}
&= \quad -\gamma U^\theta G^t + U^\theta U^R G^R + [1 + R^2 (U^\theta)^2] G^\theta + U^\theta U^z G^z + f^\theta + U^\theta U_\beta f^\beta \\
\phi : &\quad \gamma \omega_g \frac{\partial U^z}{\partial t} + \omega_g U^i \frac{\partial U^z}{\partial x^i} + \frac{\partial P_g}{\partial z} + \gamma U^z \frac{\partial P_g}{\partial t} + U^z U^i \frac{\partial P_g}{\partial x^i} \\
&= \quad -\gamma U^z G^t + U^z U^R G^R + R^2 U^z U^\theta G^\theta + [1 + (U^z)^2] G^z + f^z + U^z U_\beta f^\beta
\end{aligned}
$$

- Energy equation

$$
\begin{aligned}
- &\quad nU^t \frac{\partial}{\partial t}\left(\frac{\omega_g}{n}\right) - nU^i \frac{\partial}{\partial x^i}\left(\frac{\omega_g}{n}\right) + U^t \frac{\partial P_g}{\partial t} + U^i \frac{\partial P_g}{\partial x^i} \\
= &\quad -G^{\hat{t}}_{co} + U_\alpha f^\alpha = \Lambda_{co} - \Gamma_{co} + U_\alpha f^\alpha
\end{aligned}
$$

- Radiation energy equation

$$
\begin{aligned}
&\frac{\partial E}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(RF^R) + \frac{1}{R}\frac{\partial F^\theta}{\partial \theta} + \frac{\partial F^z}{\partial z} \\
&\quad = -G^t = \gamma\left(\Lambda_{co} - \Gamma_{co} - \underline{\bar{\chi}_{co} v_i F^i_{co}}\right)
\end{aligned}
$$

- Radiation momentum equation

$$R: \quad \frac{\partial F^R}{\partial t} + \frac{\partial P^{RR}}{\partial R} + \frac{1}{R}\frac{\partial P^{R\theta}}{\partial \theta} + \frac{\partial P^{Rz}}{\partial z} + \frac{P^{RR} - P^{\theta\theta}}{R}$$

$$= -\bar{\chi}_{co} F^R_{co} - \gamma v^R (\Gamma_{co} - \Lambda_{co}) - \frac{\gamma - 1}{v^2} v^R v_i \bar{\chi}_{co} F^i_{co}$$

$$\theta: \quad \frac{\partial F^\theta}{\partial t} + \frac{\partial P^{R\theta}}{\partial R} + \frac{1}{R}\frac{\partial P^{\theta\theta}}{\partial \theta} + \frac{\partial P^{\theta z}}{\partial z} + \frac{2P^{R\theta}}{R}$$

$$= -\bar{\chi}_{co} F^\theta_{co} - \gamma v^\theta (\Gamma_{co} - \Lambda_{co}) - \frac{\gamma - 1}{v^2} v^\theta v_i \bar{\chi}_{co} F^i_{co}$$

$$\phi: \quad \frac{\partial F^z}{\partial t} + \frac{\partial P^{Rz}}{\partial R} + \frac{1}{R}\frac{\partial P^{\theta z}}{\partial \theta} + \frac{\partial P^{zz}}{\partial z} + \frac{P^{Rz}}{R}$$

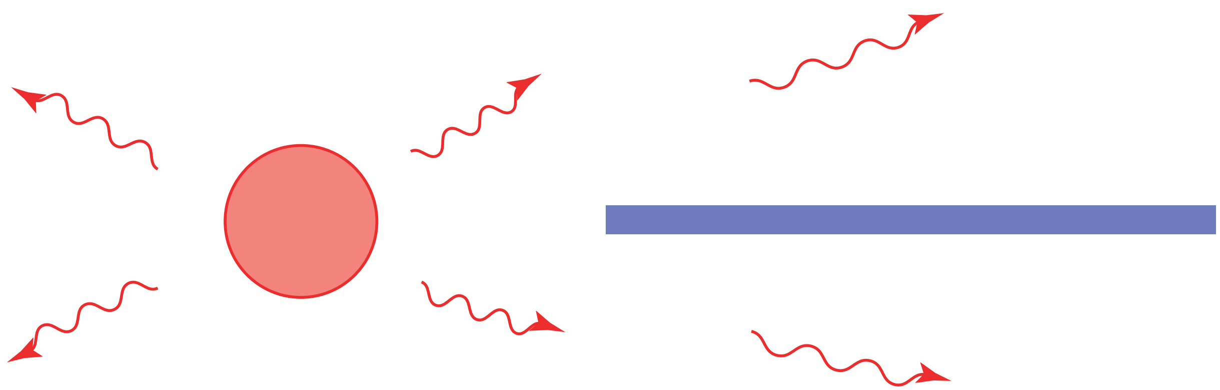
$$= -\bar{\chi}_{co} F^z_{co} - \gamma v^z (\Gamma_{co} - \Lambda_{co}) - \frac{\gamma - 1}{v^2} v^z v_i \bar{\chi}_{co} F^i_{co}$$

Example

- Spherically symmetric streaming radiation field: $E = P$

$$\boldsymbol{F} = \frac{L}{4\pi r^2}\,\hat{r} \equiv F^r\,\hat{r}$$
$$= F^R \hat{R} + F^z \hat{z}$$

with $F^R = (R/r)F^r$ and $F^z = (z/r)F^r$



- Radiation stress tensor and moments

$$R^{\hat{\alpha}\hat{\beta}} = \begin{pmatrix} E & F^R & 0 & F^z \\ F^R & \frac{R^2}{r^2}E & 0 & \frac{Rz}{r^2}E \\ 0 & 0 & 0 & 0 \\ F^z & \frac{Rz}{r^2}E & 0 & \frac{z^2}{r^2}E \end{pmatrix}$$

$$\begin{aligned} F^R_{co} &= F^R - \underline{\left[v^R (1 + \frac{R^2}{r^2}) + v^z \frac{Rz}{r^2} \right] E} \\ F^\theta_{co} &= -\underline{v^\theta E} \\ F^z_{co} &= F^z - \underline{\left[v^z (1 + \frac{z^2}{r^2}) + v^R \frac{Rz}{r^2} \right] E} \end{aligned}$$

- 1st order in $\boldsymbol{v}$

$$\gamma = 1 + O(v^2)$$
$$v^i = \tilde{u}^i + O(v^2)$$
$$\bar{\chi}_{co} = \bar{\chi} + O(v^2)$$

- Euler equations
  - R-direction

$$\omega_g \frac{\partial v^R}{\partial t} + \omega_g v^i \frac{\partial v^R}{\partial x^i} - \omega_g R (v^\theta)^2 + \frac{\partial P_g}{\partial R} + v^R \frac{\partial P_g}{\partial t} + v^R v^i \frac{\partial P_g}{\partial x^i}$$

$$= \quad f^R + \bar{\chi} F^R - \bar{\chi} \left[v^R (1 + \frac{R^2}{r^2}) + v^z \frac{Rz}{r^2} \right] E + O(v^2)$$

  - θ-direction

$$\omega_g \frac{\partial v^\theta}{\partial t} + \omega_g v^i \frac{\partial v^\theta}{\partial x^i} + 2\omega_g \frac{v^R v^\theta}{R} + \frac{1}{R} \frac{\partial P_g}{\partial \theta} + v^\theta \frac{\partial P_g}{\partial t} + v^\theta v^i \frac{\partial P_g}{\partial x^i}$$

$$= \quad f^\theta - \bar{\chi} v^\theta E + O(v^2)$$

– z-direction

$$\omega_g \frac{\partial v^z}{\partial t} + \omega_g v^i \frac{\partial v^z}{\partial x^i} + \frac{\partial P_g}{\partial z} + v^z \frac{\partial P_g}{\partial t} + v^z v^i \frac{\partial P_g}{\partial x^i}$$
$$= \quad f^z + \bar{\chi} F^z - \underline{\bar{\chi} \left[v^R \frac{Rz}{r^2} + v^z (1 + \frac{z^2}{r^2}) \right] E} + O(v^2)$$

- Energy equation

$$- \quad n \frac{\partial}{\partial t} \left(\frac{\omega_g}{n} \right) - n v^i \frac{\partial}{\partial x^i} \left(\frac{\omega_g}{n} \right) + \frac{\partial P_g}{\partial t} + v^i \frac{\partial P_g}{\partial x^i}$$
$$= \quad \Lambda_{co} - \Gamma_{co} - f^t + v_i f^i$$

Summary

- Covariant formalism of relativistic radiation hydrodynamics is not really difficult.
- Applicable to any coordinates and spacetimes.
- Easy to apply and understand.
- However,
  - frequency-integrated description
  - closure problem