Signature transition as the initial condition of the universe

Babak Vakili*
Department of Physics, Islamic Azad University, Noshahr-Chalous Branch, Chalous, Iran

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Abstract

We study the classical and quantum evolution of a universe in which the matter source is a massive Dirac spinor field and the universe is described by a spatially flat Robertson-Walker metric. We focus attention on the those classical solutions that the scale factor have smooth behavior in transition from a Euclidean to a Lorentzian domain and show that this transition happens when the cosmological constant, Λ, is negative. The resulting quantum cosmology is also studied and closed form expressions for the wave function of the universe is presented. It is shown that there is a close relationship between the quantum states and signature changing classical solutions, suggesting a mechanism for creation of a Lorentzian universe from a Euclidean region without any tunneling.

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*email: b-vakili@sbu.ac.ir

1 Introduction

An important question in cosmology is that of the initial condition from which the universe has evolved. As is well known, standard cosmological models based on classical general relativity have no suitable answer to this question. This can be traced to the fact that these models suffer from the presence of an initial singularity, the so called "Big-Bang" singularity. Any hope of dealing with such singularities would be in the development of a concomitant and conducive quantum theory of gravity. In the absence of a full theory of quantum gravity, it would be useful to describe the quantum state of the universe within the context of quantum cosmology, introduced in the works of DeWitt [1]. In this formalism which is based on the canonical quantization procedure, the evolution of universe is described by a wave function in the mini-superspace. Two major approaches in this scenario are the *tunneling* proposal first developed by Vilenkin [2]-[6], and the *no-boundary* proposal of Hartle and Hawking [7]-[9]. In the tunneling proposal the wave function is so constructed as to create a universe emerging from *nothing* by a tunneling procedure through a potential barrier in the sense of usual quantum mechanics. In the no-boundary proposal of Hartle and Hawking on the other hand, the wave function is constructed by a path integral over all compact Euclidean 4-manifolds. A problem related to this approach is that of signature transition from a Euclidean to a Lorentzian manifold. The notion of signature transition was first addressed in [7], where the authors argued that in quantum cosmology amplitudes for gravity should be expressed as the sum of all compact Riemannian manifolds whose boundaries are located at the signature changing hypersurface. In traditional point of view, a feature in general relativity is that one usually fixes the signature of the space-time metric before trying to solve Einstein's field equations. However there is no *a priori* reason for doing so and it is well known that the field equations do not demand this property, that is, if one relaxes this condition one may find solutions to the field equations which when parameterized suitably, can either have Euclidean or Lorentzian signature [11]-[15].

Here we deal with the classical and quantum cosmology of a model in which a massive free spinor field is coupled to gravity in a spatially flat Robertson-Walker (RW) space-time. From the classical solutions of the resulting Einstein-Dirac system we have chosen those that exhibit a smooth transition from Euclidean to a Lorentzian region, *i.e.*, signature changing solutions. Although in classical gravity such solutions may not seem to be too interesting, we show that there is a close relationship between these solutions and the those resulting from the Wheeler-DeWitt (WD) equation in the corresponding quantum cosmology. We show that these solutions predict creation of a universe by a continuous transition from a classically forbidden (Euclidean) to a classically allowed (Lorentzian) domain and that they are in agreement with the classical signature changing solutions.

2 The model

We start with a space-time metric of the form

$$ds^2 = -dt^2 + R^2(t)(dx^2 + dy^2 + dz^2), \tag{1}$$

which describe a spatially flat RW universe with scale factor $R(t)$. The scalar curvature corresponding to metric (1) is

$$\mathcal{R} = 6\left[\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2}\right], \tag{2}$$

where a dot represents differentiation with respect to t. We may parameterize the metric in such a way as to allow the Euclidean signature $(+,+,+,+)$ becoming Lorentzian $(-,+,+,+)$ [11]-[15]. To this and other ends, we parameterize the metric as in [14] and [15] by adapting the chart $\{\beta, x, y, z\}$ where the hypersurface of signature change would be characterized by $\beta = 0$. The metric can then be parameterized in terms of the scale factor $R(\beta)$ and laps function β to take the form

$$ds^2 = -\beta d\beta^2 + R^2(\beta)(dx^2 + dy^2 + dz^2). \tag{3}$$

It is now clear that the sign of the laps function β determines the signature of the metric, being Lorentzian if $\beta > 0$ and Euclidean if $\beta < 0$. For the Lorentzian region the traditional cosmic time can be recovered by the substitution $t = \frac{2}{3}\beta^{3/2}$. Also, adapting the chart $\{t, x, y, z\}$ in this region we shall write any dynamical field such as the scale factor or matter fields as $\Phi(t) = \Phi(\beta(t))$. As a result of the above discussion, we see that the signature changing hypersurface divides the manifold into two Euclidean and Lorentzian domains. From the point of view of the Einstein field equations, these two regions are classically forbidden and classically allowed solutions of the gravitational field equations respectively. We formulate our differential equations in a region which does not include $\beta = 0$ and seek solutions for any dynamical field that smoothly passes through $\beta = 0$ hypersurface. These solutions are called signature changing solutions and we shall see that they are classical description of the quantum cosmological states of the model. Indeed, we will encounter the Euclidean and Lorentzian regions again when dealing with the solutions of the WD equation later on.

To construct the field equations, let us start with the action

$$\mathcal{S} = \int (L_{grav} + L_{matt})\sqrt{-g}d^4x, \tag{4}$$

where

$$L_{grav} = \mathcal{R} - \Lambda, \tag{5}$$

is the Einstein-Hilbert Lagrangian for the gravitational field with cosmological constant Λ, and L_{matt} represents the Lagrangian of the matter source, which we assume to be a massive spinor field. As is well known, the Dirac equation describing dynamics of a spinor field ψ can be obtained from the Lagrangian

$$L_{matt} = \frac{1}{2}\left[\bar{\psi}\gamma^\mu(\partial_\mu + \Gamma_\mu)\psi - \bar{\psi}(\overleftarrow{\partial_\mu} - \Gamma_\mu)\gamma^\mu\psi\right] - V(\bar{\psi}, \psi), \tag{6}$$

where γ^μ are the Dirac matrices associated with the space-time metric satisfying the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, Γ_μ are spin connections and $V(\bar{\psi}, \psi)$ is a potential describing the interaction of the spinor field with itself. In the case of a free spinor field of mass m we have $V = m\bar{\psi}\psi$. The γ^μ matrices are related to the flat Dirac matrices, γ^a, through the tetrads e^a_μ as follows

$$\gamma^\mu = e^\mu_a\gamma^a, \qquad \gamma_\mu = e^a_\mu\gamma_a. \tag{7}$$

For metric (1) the tetrads can be easily obtained from their definition, that is, $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$, leading to

$$e^a_\mu = diag(1, R, R, R), \qquad e^\mu_a = diag(1, 1/R, 1/R, 1/R). \tag{8}$$

Also, the spin connections satisfy the relation

$$\Gamma_\mu = \frac{1}{4}g_{\nu\lambda}(\partial_\mu e^\lambda_a + \Gamma^\lambda_{\sigma\mu}e^\sigma_a)\gamma^\nu\gamma^a. \tag{9}$$

Thus, for the line element (1) use of (7) and (8) yields

$$\Gamma_0 = 0, \qquad \Gamma_i = -\frac{\dot{R}}{2}\gamma^0\gamma^i. \tag{10}$$

Here, γ^0 and γ^i are the Dirac matrices in Minkowski space and we have adapted the following representation [16]

$$\gamma^0 = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}. \tag{11}$$

The preliminary set-up for writing the action is now complete. By substituting the above results into (3), considering a homogeneous spinor field $\psi = \psi(t)$, and integrating over spatial dimensions, we are led to an effective Lagrangian in the mini-superspace $\{R, \bar{\psi}, \psi\}$ as follows

$$\mathcal{L} = 3R\dot{R}^2 + \Lambda R^3 + \frac{1}{2}R^3\left[\bar{\psi}\gamma^0\dot{\psi} - \dot{\bar{\psi}}\gamma^0\psi - 2V(\bar{\psi}, \psi)\right]. \tag{12}$$

3 Field equations and classical solutions

Variation of Lagrangian (12) with respect to $\bar{\psi}$, ψ and R yields the equation of motion for the spinor and gravitational fields respectively

$$\dot{\psi} + \frac{3}{2}\frac{\dot{R}}{R}\psi + \gamma^0 \frac{\partial V}{\partial \bar{\psi}} = 0, \tag{13}$$

$$\dot{\bar{\psi}} + \frac{3}{2}\frac{\dot{R}}{R}\bar{\psi} - \frac{\partial V}{\partial \psi}\gamma^0 = 0, \tag{14}$$

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} - \Lambda = \frac{1}{2}\left(\bar{\psi}\frac{\partial V}{\partial \bar{\psi}} + \frac{\partial V}{\partial \psi}\psi\right) - V(\bar{\psi}, \psi). \tag{15}$$

Also, we have the "zero energy" condition given by

$$\mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{R}}\dot{R} + \frac{\partial \mathcal{L}}{\partial \dot{\psi}}\dot{\psi} + \dot{\bar{\psi}}\frac{\partial \mathcal{L}}{\partial \dot{\bar{\psi}}} - \mathcal{L} = 0, \tag{16}$$

which yields the constraint equation

$$3\frac{\dot{R}^2}{R^2} - \Lambda = -V(\bar{\psi}, \psi). \tag{17}$$

Integrability of the above Einstein-Dirac equations depends on the choice of a suitable form for $V(\bar{\psi}, \psi)$. This potential which describe a physical self-interacting spinor field, is usually an invariant function constructed from the spinor field ψ and its adjoint $\bar{\psi}$. Some of the common forms for V are: $V = m\bar{\psi}\psi$ representing a free spinor field of mass m, $V = m\bar{\psi}\psi + J^\mu J_\mu$ where $J^\mu = \bar{\psi}\gamma^\mu\psi$ known as the Thirring model, $V = m\bar{\psi}\psi + (\bar{\psi}\psi)^2$ called the Gross-Neveu model and $V = m\bar{\psi}\psi + (\bar{\psi}\psi)^2 - (\bar{\psi}\gamma^5\psi)^2$ also known as the chiral Gross-Neveu model. Since here we are going to study signature transition for the solutions of the problem at hand, we concentrate the simplest form for V that is a free spinor field $V = m\bar{\psi}\psi$. As we will see, this choice of potential either give the exact solutions of the field equations or show signature transition effect. In this case the field equations (13)-(15) can easily obtained and the results are depended of the sign of the cosmological constant Λ. In the case of a zero cosmological constant the scale factor in terms of β is

$$R(\beta) = (M\beta^{3/2} + A)^{2/3}, \tag{18}$$

also, the scalar curvature and the energy density $\rho = T_{00} = -V(\bar{\psi}, \psi)$ of the spinor field are found to be

$$\mathcal{R} = \frac{3M^2}{(M\beta^{3/2} + A)^2}, \qquad \rho = \frac{3M^2}{(M\beta^{3/2} + A)^2}, \tag{19}$$

where M (being proportional to m) and A are integration constants. We see that if $A \neq 0$, the scale factor $R(\beta)$ is not a real function in the Euclidean region $\beta < 0$. If $A = 0$, then $R(\beta) \sim \beta$ is an unbounded function in both Euclidean and Lorentzian regions, passing continuously through $\beta = 0$. However, in this case the functions $\mathcal{R}$ and ρ both have a singular behavior in $\beta = 0$ but are well defined in the domains $\beta > 0$ and $\beta < 0$. The solutions in the case of zero cosmological constant are therefore not suitable candidates for exhibiting signature transition behavior. In the case of a negative cosmological constant $\Lambda < 0$, the solutions of the system (13)-(15) can be written in terms of the evolution parameter β as

$$R(\beta) = \left(\frac{M}{-\Lambda}\right)^{1/3} \cos^{2/3}\left(\frac{\sqrt{-3\Lambda}}{3}\beta^{3/2}\right), \tag{20}$$

$$\mathcal{R} = -\Lambda\left[\tan^2\left(\frac{\sqrt{-3\Lambda}}{3}\beta^{3/2}\right) - 3\right], \qquad \rho = \frac{-\Lambda}{\cos^2\left((\sqrt{-3\Lambda}/3)\beta^{3/2}\right)}, \tag{21}$$

where M is an integration constant proportional to m, the mass of the spinor field. The other integration constant is chosen so that $\dot{R}(\beta=0)=0$. The scale factor $R(\beta)$ in this case is a real function and behaves exponentially for $\beta<0$, passing smoothly through $\beta=0$, and becomes a bounded oscillatory function for $\beta>0$. Also, the physical quantities like $\mathcal{R}$ and ρ are regular functions both in the Euclidean and Lorentzian domains and pass continuously through the $\beta=0$ hypersurface. Thus all of the above results show that the solutions in the case of a negative cosmological constant exhibit signature transition from a Euclidean (where in which there is no time) to a Lorentzian domain. Finally, if $\Lambda>0$, the solutions can easily be obtained by the replacement of the" cos " function in (20) with its hyperbolic counterpart and a quick look at the resulting solutions shows that they do not exhibit the signature changing behavior. Thus, in summary, the above discussion shows that within the context of our model, a universe with negative cosmological constant can undergo signature transition from a classically forbidden Euclidean to a classically allowed Lorentzian domain through the $\beta=0$ hypersurface, where its matter source is a spinor field. This issue will become more clear when we study the quantum cosmology of the model in the next section.

4 Quantization of the model

The study of quantum cosmology of the model presented above is the goal we shall pursue in this section. For this purpose we construct the Hamiltonian of our model. To simplify the Lagrangian (12), consider the change of variable $u=R^{3/2}$. In terms of this new variable Lagrangian (12) takes the form

$$\mathcal{L}=\frac{4}{3}\dot{u}^2+\Lambda u^2+\frac{1}{2}u^2\left[\bar{\psi}\gamma^0\dot{\psi}-\dot{\bar{\psi}}\gamma^0\psi-2V(\bar{\psi},\psi)\right]. \quad (22)$$

The momenta conjugate to our dynamical variables are as

$$p_u=\frac{\partial\mathcal{L}}{\partial\dot{u}}=\frac{8}{3}\dot{u}, \quad p_\psi=\frac{\partial\mathcal{L}}{\partial\dot{\psi}}=\frac{1}{2}u^2\bar{\psi}\gamma^0, \quad p_{\bar{\psi}}=\frac{\partial\mathcal{L}}{\partial\dot{\bar{\psi}}}=-\frac{1}{2}u^2\gamma^0\psi. \quad (23)$$

In terms of conjugate momenta, the Hamiltonian is given by

$$\mathcal{H}=p_u\dot{u}+p_\psi\dot{\psi}+\dot{\bar{\psi}}p_{\bar{\psi}}-\mathcal{L}=0, \quad (24)$$

with the result

$$\mathcal{H}=u^2p_u^2-\frac{16}{3}\Lambda u^4+\frac{32}{3}mp_\psi p_{\bar{\psi}}=0, \quad (25)$$

in which we assumed that the spinor field is a free with mass m, *i.e.*, $V=m\bar{\psi}\psi$ and also use the relation $\bar{\psi}\psi=2u^{-4}p_\psi p_{\bar{\psi}}$. To quantize the dynamical variables of the model, we use the WD equation, that is, $\mathcal{H}\Psi=0$, where $\mathcal{H}$ is the operator form of the above Hamiltonian and Ψ is the wave function of the universe, a function of the scale factor and the matter field. To guarantee Hermiticity, the operator form corresponding to this Hamiltonian should be written as

$$\mathcal{H}=u^p p_u u^{2-2p}p_u u^p+\omega^2u^4+Mp_\psi p_{\bar{\psi}}=0, \quad (26)$$

where the parameter p denotes the ambiguity in the ordering of factors u and p_u in the first term of (25), $\omega^2=-\frac{16}{3}\Lambda$ and $M=\frac{32}{3}m$. With the replacement $p_u\rightarrow -i\frac{\partial}{\partial u}$ and similarly for p_ψ and $p_{\bar{\psi}}$ and taking $p=0$, the WD equation reads

$$\left(-\frac{\partial}{\partial u}u^2\frac{\partial}{\partial u}+\omega^2u^4-M\frac{\partial^2}{\partial\psi\partial\bar{\psi}}\right)\Psi(u,\bar{\psi}\psi)=0. \quad (27)$$

The solutions of the above differential equation are separable into the form $\Psi(u,\bar{\psi}\psi)=U(u)f(\bar{\psi}\psi)$ leading to

$$\frac{1}{f}\frac{\partial^2 f}{\partial\psi\partial\bar{\psi}}=\alpha^2, \quad (28)$$

$$\frac{d^2U}{du^2}+\frac{2}{u}\frac{dU}{du}+\left(\frac{M\alpha^2}{u^2}-\omega^2u^2\right)U=0, \tag{29}$$

where α^2 is a separation constant. To find the solutions of equation (28) we use the ansatz

$$f(\bar{\psi}\psi)=\sum_{n=0}^{\infty}c_n(\bar{\psi}\psi)^n. \tag{30}$$

After a little algebra we find $c_n=\frac{\alpha^{2n}}{(n!)^2}c_0$ and thus

$$f(\bar{\psi}\psi)=\sum_{n=0}^{\infty}\frac{\alpha^{2n}}{(n!)^2}(\bar{\psi}\psi)^n. \tag{31}$$

It is easy to check that this series converge for all values of $\bar{\psi}\psi$. On the other hand equation (29) is a Schrödinger-like equation for a particle with zero energy moving in the field of the effective potential

$$\mathcal{U}(u)=\omega^2u^2-\frac{M\alpha^2}{u^2}. \tag{32}$$

For a negative cosmological constant ($\omega^2>0$) this potential has an infinity deep minimum at $u=o$. In the presence of this potential the mini-superspace can be divided into two regions $\mathcal{U}>0$ and $\mathcal{U}<0$ which could be termed as the classically forbidden or Euclidean and classically allowed or Lorentzian regions respectively. The boundary between the two regions is given by $\mathcal{U}=0$, that is at

$$u^2=R^3=\frac{M\alpha}{\omega}\sim\frac{M}{-\Lambda}\Rightarrow R=\left(\frac{M}{-\Lambda}\right)^{1/3} \tag{33}$$

Comparison of the above value of R and classical solution (20) suggests that it corresponds to $\beta=0$. Thus, the same boundary separates the Euclidean and Lorentzian regions in both classical and quantum solutions. In the Euclidean domain the wave function has a exponentially behavior and in the Lorentzian region we have the wave function of oscillatory nature [3, 4]. Now let us deal with the solution of equation (29). The solution of this equation can be written in terms of modified Bessel functions

$$U(u)=u^{-1/2}K_\nu(\omega u^2),\quad \textit{and}\quad U(u)=u^{-1/2}I_\nu(\omega u^2), \tag{34}$$

where $\nu^2=1/16-M\alpha^2/4$. To satisfy $\Psi(u\to\infty)=0$ we restrict ourselves to function $K_\nu(x)$. Thus for a negative cosmological constant the eigenfunctions of the WD equation can be written as

$$\Psi_n(u,\bar{\psi}\psi)=u^{-1/2}K_\nu(\omega u^2)\frac{\alpha^{2n}}{(n!)^2}(\bar{\psi}\psi)^n. \tag{35}$$

To avoid singularity at $u=0$ the order of the function $K_\nu(x)$ should be Pure imaginary or zero; $\nu^2\leq 0$ [17], which results in the interval $M\geq 1/4\alpha^2$ for the mass (or energy) of the spinor field. The general solution of the WD equation can then written as

$$\Psi(u,\bar{\psi})=\sum_n c_n\Psi_n(u,\bar{\psi}\psi). \tag{36}$$

To have an exponential wave function in the Euclidean domain we must take $n=0$ in (36), *i.e.* the matter must be in its ground state. Taking more terms in (36) leads us to the wave function in the Lorentzian region. Summarizing, we have the following wave functions

$$\Psi_E(u,\bar{\psi}\psi)=u^{-1/2}K_0(\omega u^2), \tag{37}$$

in the Euclidean region and

$$\Psi_L(u,\bar{\psi}\psi)=u^{-1/2}K_\nu(\omega u^2)\sum_{n=1}^{\infty}\frac{\alpha^{2n}}{(n!)^2}(\bar{\psi}\psi)^n, \tag{38}$$

with $\nu^2 < 0$ in Lorentzian region. The wave functions (37) and (38) describe a universe emerging out of the Euclidean region with a smoothly changing signature and correspond to the signature changing classical solutions. The creation of the Lorentzian universe in this scenario is comparable to the quantum tunneling from *nothing* in the Vilenkin's proposal [2]-[6] where nothing is a 3-manifold of vanishing size. However, note that the potential (32) has no maximum and therefore should not be considered as a potential barrier like those described in [2]-[6]. Summarizing, the creation of a Lorentzian universe from a Euclidean one, is through a smoothly signature transition without any tunneling, indeed the excitation of the matter states evolve the universe from the Euclidean domain.

In the case of a positive cosmological constant, potential (32) is negative and thus does not divide the mini-superspace into two Euclidean and Lorentzian regions. Therefore, there is no mechanism for a smooth transition from the classically forbidden to the classically allowed regions and the creation of a Lorentzian universe would not be possible. We have seen in section 3 that the classical solutions with $\Lambda > 0$ do not exhibit any signature transition properties, that is, the classical and quantum solutions are in agreement in this case as well.

5 Conclusions

In this work we have studied the classical and quantum evolution of the Einstein-Dirac system in a spatially flat RW universe. From the classical solutions of this system we have chosen those that admit a degenerate metric for which the scale factor of the universe has a continuous behavior in passing from a classically forbidden (Euclidean) to a classically allowed (Lorentzian) region. We have shown that this happens when the cosmological constant is negative. The quantum cosmology of the model presented above and the ensuing WD equation is amenable to exact solutions in terms of Bessel functions. We have found that the Euclidean and Lorentzian regions also happens at the quantum level. In the Euclidean domain the wave function behaves exponentially, and with excitation of the matter from its ground state the universe undergo the Lorentzian region with the oscillatory wave function. Within the context of this model the creation of the universe is described by a smooth transition from a Euclidean to a Lorentzian region without any tunneling.

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